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Opérateurs sans multiplicité

Par BÉLA SZ.-NAGY à Szeged et CIPRIAN FOIAŞ à Bucarest

Hommage à Monsieur S. Mazur

Introduction

Dans la monographie [A], n° IX. 3, on a montré que certains aspects de la théorie des matrices finies peuvent être retrouvés dans l'étude de certaines classes d'opérateurs de l'espace de Hilbert. Le but de cette Note est de poursuivre la recherche de ces analogies.

On s'occupera en particulier d'opérateurs T „sans multiplicité". L'une des caractérisations de ces opérateurs est qu'il n'y a pas de sous-espaces invariants $\mathfrak{L}_1, \mathfrak{L}_2$ ($\mathfrak{L}_1 \neq \mathfrak{L}_2$) tels que les restrictions $T|\mathfrak{L}_1$ et $T|\mathfrak{L}_2$ soient quasi-similaires. (Pour la définition de la quasi-similitude, cf. [A].)

Nous montrerons que la plupart des caractérisations qu'on obtient pour tels opérateurs dans les espaces de dimension finie, gardent leur validité aussi pour les opérateurs des classes $C_0(N)$ ($N=1, 2, \cdots$) dans l'espace de Hilbert (de dimension finie ou infinie), si l'on remplace la fonction matricielle $T-\lambda I$ et le polynôme minimum par la fonction caractéristique $\Theta_T(\lambda)$ et la fonction intérieure minimum, selon les cas.

Il est manifeste que pour les opérateurs dans les espaces de dimension finie la notion de la quasi-similitude coïncide avec celle de la similitude. Nous montrerons au n° 8 que pour les opérateurs des classes $C_0(N)$ cela n'est plus le cas.

Observons que de tout opérateur T dans l'espace euclidien E^N de dimension finie N on obtient un opérateur $T'=\alpha T$ de classe $C_0(N)$ si l'on choisit le facteur numérique $\alpha \neq 0$ tel que $\|T'\|<1$. Ainsi, les résultats sur les opérateurs de classe $C_0(N)$ s'appliquent aussi aux opérateurs dans E^N. Dans ce cas la fonction minimum de T' sera un produit fini de Blaschke; le produit des numérateurs de ce produit fournit alors le polynôme minimum de T', d'où le polynôme minimum de T résulte d'une manière évidente. Nous laissons au lecteur de se rendre compte des détails de cette réduction. Nous préférons de rechercher les opérateurs dans des espaces de dimension finie d'une manière directe, en nous appuyant sur les méthodes classiques de la théorie des matrices finies. Ce sera l'objet du n° 1. Les opérateurs de classe $C_0(N)$ seront étudiés dans les numéros ultérieurs, sauf le n° 3 qui contient un lemme sur les fonctions intérieures.

1. Matrices finies

1. Soit T une matrice finie de type de Jordan:

$$T = (\lambda_1 I_{m_1} + J_{m_1}) \dotplus \cdots \dotplus (\lambda_r I_{m_r} + J_{m_r}) \tag{1.1}$$

où I_m désigne la matrice unité et J_m désigne la matrice auxiliaire, de rang m.[1] Considéré comme opérateur dans l'espace euclidien complexe E^N de dimension $N = m_1 + \cdots + m_r$, T a les valeurs propres $\lambda_1, \cdots, \lambda_r$, comptées conformément à leurs multiplicités.

Le déterminant de $\lambda I_N - T$ est égal à

$$D_T(\lambda) = \prod_1^r (\lambda - \lambda_i)^{m_i},$$

tandis que le polynôme minimum de T est

$$M_T(\lambda) = \prod_1^s (\lambda - \mu_k)^{n_k}$$

où μ_k parcourt les valeurs propres différentes de T et l'exposant n_k est égal au maximum des exposants m_i correspondant aux valeurs λ_i égales à μ_k.

Pour qu'on ait $D_T = M_T$ il faut et il suffit donc que toutes les valeurs propres de T soient *simples*.

Observons aussi que, dans le cas général, le polynôme minimum de T est égal à celui de la matrice

$$T' = (\mu_1 I_{n_1} + J_{n_1}) \dotplus \cdots \dotplus (\mu_s I_{n_s} + J_{n_s})$$

dont les valeurs propres sont simples. Regardé comme opérateur, T' peut être considéré comme la restriction de T à un sous-espace invariant. Il y a éventuellement plusieurs sous-espaces invariants de ce type, par exemple dans le cas où il y a, parmi les nombres m_i correspondant aux λ_i égaux à la même valeur μ_k, plusieurs qui atteignent le maximum n_k.

2. Cherchons des conditions pour qu'un vecteur x soit cyclique pour T, c'est-à-dire que $x, Tx, T^2x, \cdots$ sous-tendent l'espace E^N.

A cette fin, observons d'abord que pour un polynôme $p(\lambda)$ quelconque on a

$$p(T) = \underset{1 \leq i \leq r}{\dotplus} \left[p(\lambda_i) I_{m_i} + \frac{1}{1!} p'(\lambda_i) J_{m_i} + \cdots + \frac{1}{(m_i - 1)!} p^{(m_i - 1)}(\lambda_i) J_{m_i}^{m_i - 1} \right]. \tag{1.2}$$

Numérotons les composantes des vecteurs $x \in E^N$ conformément à la décomposition en somme directe (1. 1), par deux indices:

$$x = (x_{11}, \ldots, x_{1m_1}; x_{21}, \ldots, x_{2m_2}; \ldots; x_{r1}, \ldots, x_{rm_r}).$$

---

[1] $J_m = (a_{ik})$ $(i, k = 1, \ldots, m)$, avec $a_{i,i+1} = 1$ pour $i = 1, \ldots, m-1$ et $a_{ik} = 0$ pour les autres pairs i, k.

En vertu de (1. 2) on a alors pour $y=p(T)x$:

$$
\left\{
\begin{aligned}
y_{i1} &= p(\lambda_i)x_{i1}+\frac{1}{1!}p'(\lambda_i)x_{i2}+\cdots \quad \cdots+\frac{1}{(m_i-1)!}\,p^{(m_i-1)}(\lambda_i)x_{im_i},\\
y_{i2} &= \qquad\qquad\qquad p(\lambda_i)x_{i2}+\cdots \quad \cdots+\frac{1}{(m_i-2)!}\,p^{(m_i-2)}(\lambda_i)x_{im_i},\\
&\vdots\\
y_{im_i} &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad p(\lambda_i)x_{im_i}
\end{aligned}
\right. \tag{1. 3}
$$

$(i=1, \cdots, r)$. Lorsque $x_{im_i}=0$ pour un i, on a donc $y_{im_i}=0$ et cela pour tout polynôme p. On en déduit que si x est un vecteur *cyclique* pour T, on a nécessairement $x_{im_i}\neq 0$ pour $i=1, \cdots, r$. Cela entraîne, à son tour, que les valeurs propres de T sont toutes simples. En effet, si $1\leqq a<b\leqq r$, le vecteur $z\in E^N$ défini par

$$z_{am_a}=x_{bm_b}, \quad z_{bm_b}=-x_{am_a}, \quad z_{ij}=0 \quad \text{pour les autres} \quad i, j,$$

est différent de 0 et pour $y=p(T)x$ on a

$$(y, z)=p(\lambda_a)x_{am_a}\cdot\overline{x_{bm_b}}+p(\lambda_b)x_{bm_b}\cdot\left(\overline{-x_{am_a}}\right)=[p(\lambda_a)-p(\lambda_b)]x_{am_a}\overline{x_{bm_b}};$$

si l'on avait $\lambda_a=\lambda_b$, z serait donc orthogonal à $p(T)x$ quel que soit p, ce qui contredirait l'hypothèse que x est cyclique. Donc $\lambda_a\neq\lambda_b$.

Inversement, si les valeurs propres de T sont simples, tout vecteur $x\in E^N$ tel que $x_{im_i}\neq 0$ $(i=1, \cdots, r)$, est cyclique pour T. Cela veut dire que, x étant fixé de cette façon, pour tout vecteur $y\in E^N$ il existe un polynôme p vérifiant le système d'équations (1. 3). Or, ce système se réduit évidemment à un système de la forme

$$p(\lambda_i)=t_{i1},\ p'(\lambda_i)=t_{i2}, \ldots, p^{(m_i-1)}(\lambda_i)=t_{im_i} \qquad (i=1, \ldots, r),$$

les valeurs t_{ij} dérivant des composantes des vecteurs x et y. Il s'agit donc d'un problème d'interpolation d'Hermite, et ce problème a pour solution un polynôme p, même de degré $\leqq N-1$.

Donc, une condition nécessaire et suffisante pour que T admette un vecteur cyclique, est que les valeurs propres de T soient simples.

De plus, s'il existe un vecteur cyclique pour T, tous les vecteurs x sont cycliques sauf peut-être les vecteurs situés dans un nombre fini $(\leqq N)$ de sous-espaces de E^N, de dimension $N-1$; ainsi, l'ensemble des vecteurs cycliques est ou bien vide ou bien dense dans E^N.

3. En supposant que les valeurs propres de T sont simples (c'est-à-dire $\lambda_1, \cdots, \lambda_r$ des valeurs différentes), cherchons de déterminer les sous-espaces invariants pour T. Soit x un vecteur fixé dans E^N. Si les composantes $x_{i1}, \cdots, x_{im_i}$ ne sont pas toutes égales à 0, soit x_{ik_i} la dernière entre elles qui n'est pas 0; autrement on pose $k_i=0$. En se servant toujours de l'interpolation d'Hermite on déduit des équations (1. 3) que si p parcourt les polynômes, $y=p(T)x$ parcourt les vecteurs dont les composantes

$$y_{ij} \quad (k_i < j \leqq m_i; \quad i = 1, \ldots, r)$$

sont 0 et les autres arbitraires. L'ensemble de ces y est un sous-espace de E^N qui est évidemment déterminé par les nombres $k_1, \cdots, k_r$; désignons-le par $\mathfrak{L}(k_1, \cdots, k_r)$. En faisant x varier, on conclut aussitôt que les sous-espaces invariants pour T sont précisément les sous-espaces $\mathfrak{L}(k_1, \cdots, k_r)$ où les nombres k_i peuvent être préscrits arbitrairement sous la condition $0 \leqq k_i \leqq m_i$ $(i=1, \cdots, r)$.

Le polynôme minimum de la restriction de l'opérateur T à $\mathfrak{L}(k_1, \cdots, k_r)$, qui doit être un diviseur de $M_T(\lambda)$, est évidemment égal à

$$M(\lambda) = \prod_1^s (\lambda - \lambda_i)^{k_i}.$$

Ainsi, si les valeurs propres de T sont simples, il y a une correspondance biunivoque entre les diviseurs $M(\lambda)$ de $M_T(\lambda)$ et les sous-espaces invariants pour T. Notamment, pour tout diviseur $M(\lambda)$ il existe un sous-espace invariant $\mathfrak{L}$ et un seul tel que la restriction de T à $\mathfrak{L}$ ait le polynôme minimum $M(\lambda)$; notamment

$$\mathfrak{L} = \{x: M(T)x = 0\}.$$

Inversement, cette propriété caractérise les opérateurs T dont toutes les valeurs propres sont simples. Il suffit même de savoir qu'il n'existe pas de sous-espace non-banal $\mathfrak{L}$, invariant pour T, tel que $T|\mathfrak{L}$ ait le même polynôme minimum que T.

A cet effet, rappelons le fait, observé dans le paragraphe 1, que pour tout T il existe un sous-espace invariant $\mathfrak{L}'$ tel que $T' = T|\mathfrak{L}'$ ait le même polynôme minimum que T et que les valeurs propres de T' soient simples. Sous l'hypothèse faite sur T, $\mathfrak{L}'$ ne peut être différent de l'espace entier, donc $T' = T$ et par conséquent T a ses valeurs propres simples.

4. Deux opérateurs similaires dans des espaces de dimension finie ont évidemment le même polynôme minimum. Il s'ensuit que si T jouit de la propriété que ses restrictions à des sous-espaces invariants différents ont des polynômes minimum différents, alors T jouit aussi de la propriété que ses restrictions à des sous-espaces différents ne sont pas similaires.

Inversement, cette dernière propriété entraîne que les valeurs propres de T sont simples et alors aussi la première propriété. En effet, si x_a, x_b étaient des vecteurs propres linéairement indépendants de T, correspondant à la même valeur propre, les sous-espaces unidimensionnels $\{cx_a\}$, $\{cx_b\}$ (c complexe) seraient différents, mais les restrictions de T à ces sous-espaces seraient similaires (même unitairement équivalentes).

5. Observons encore que si le vecteur x_0 est cyclique pour T et si X est un opérateur permutant à T, Xx_0 est la combinaison linéaire d'un nombre fini des vecteurs $T^i x_0$ $(i=0, 1, \cdots)$, soit $Xx_0 = a_0 x_0 + a_1 Tx_0 + \cdots + a_\nu T^\nu x_0$, d'où $XT^i x_0 = T^i X x_0 = (a_0 I + a_1 T + \cdots + a_\nu T^\nu) T^i x_0$ et par conséquent $X = q(T)$ avec $q(\lambda) = a_0 + a_1 \lambda + \cdots + a_\nu \lambda^\nu$: X est un polynôme de T.

Inversement, si T jouit de la propriété que tout opérateur X tel que $TX = XT$, est un polynôme de T, alors les valeurs propres de T sont simples (et par conséquent il existe un vecteur cyclique). En effet, soit E_a la somme directe, analogue à celle fournissant T (cf. (1. 1)), dont le a-ième terme est égal à I_{m_a} et les autres égaux aux matrices O de rang correspondant. E_a permute à T, mais si T a des valeurs propres multiples, p.ex. si $\lambda_a = \lambda_b$ $(a \neq b)$, E_a n'est pas de la forme $p(T)$, parce que les éléments diagonaux de $p(T)$ sont égaux à la même valeur $p(\lambda_a)$ dans la a-ième et dans la b-ième cellule.

6. Comme tout opérateur T dans un espace $\mathfrak{H}$ de dimension finie a, par rapport à une base convenable, la matrice de type (1. 1), on peut résumer les résultats obtenus comme il suit:

Théorème I. *Pour tout opérateur T dans un espace de dimension finie $\mathfrak{H}$, il existe un sous-espace invariant $\mathfrak{H}_0$ tel que les valeurs propres de $T_0 = T|\mathfrak{H}_0$ sont toutes simples et que M_{T_0} soit égal à M_T.*

Théorème II. *Pour un opérateur T dans un espace $\mathfrak{H}$ de dimension finie les conditions suivantes sont équivalentes:*

(o) *toutes les valeurs propres de T sont simples;*

(i) *il existe un vecteur cyclique pour T;*

(ii) *le polynôme minimum de T est égal au déterminant de $\lambda I - T$ par rapport à une base quelconque dans $\mathfrak{H}$: $M_T(\lambda) = D_T(\lambda)$;*

(iii) *pour tout diviseur M du polynôme minimum M_T il existe un sous-espace $\mathfrak{L}$ invariant pour T et un seul tel que le polynôme minimum de $T|\mathfrak{L}$ soit égal à M, notamment le sous-espace $\mathfrak{L} = \mathfrak{H}_M$ où*

$$\mathfrak{H}_M = \{x \colon x \in \mathfrak{H},\ M(T)x = 0\};$$

(iv) *il n'y a pas de sous-espace propre $\mathfrak{L}$ de $\mathfrak{H}$, invariant pour T, tel que $M_{T|\mathfrak{L}}$ soit égal à M;*

(v) *il n'y a pas des sous-espaces $\mathfrak{L}_1$, $\mathfrak{L}_2$ $(\mathfrak{L}_1 \neq \mathfrak{L}_2)$ invariants pour T, tels que $T|\mathfrak{L}_1$ et $T|\mathfrak{L}_2$ soient similaires;*

(vi) *tout opérateur X permutant à T est un polynôme de T: $X = q(T)$.*

Définition. Les opérateurs T dans des espaces de dimension finie, vérifiant les conditions équivalentes (o)—(vi), seront appelés *sans multiplicité*.

Théorème III. *Pour tout opérateur T dans un espace de dimension finie $\mathfrak{H}$, l'ensemble des vecteurs cycliques est ou bien vide, ou bien dense dans $\mathfrak{H}$.*

Pour terminer ajoutons la remarque suivante, conséquence immédiate de la forme de Jordan des matrices: *si T_1 et T_2 sont sans multiplicité, ils sont similaires si leurs polynômes minimum coïncident, et dans ce cas seulement.*

2. Opérateurs de classe $C_0(N)$. Théorèmes et corollaires

Dans la suite nous cherchons les analogues des propriétés établies ci-dessus, pour certains opérateurs dans des espaces de Hilbert $\mathfrak{H}$ (de dimension finie ou infinie). Nous allons envisager notamment les classes $C_0(N)$ $(N=0, 1, \cdots)$ composées des opérateurs dans $\mathfrak{H}$ tels que

$$\|T\| \leqq 1, \quad T^n \to O \quad \text{et} \quad T^{*n} \to O \quad (n \to \infty)$$

(c'est-à-dire $T \in C_{00}$) et que les sous-espaces de défaut

$$\mathfrak{D}_T = \overline{D_T \mathfrak{H}} \quad \text{et} \quad \mathfrak{D}_{T^*} = \overline{D_{T^*} \mathfrak{H}}$$

sont de dimension N; D_T et D_{T^*} désignent les opérateurs de défaut

$$D_T = (I - T^* T)^{1/2}, \quad D_{T^*} = (I - TT^*)^{1/2}. \quad ^{2})$$

Rappelons quelques faits sur les opérateurs de classe $C_0(N)$, établis dans [A].

$C_0(N)$ est comprise dans la classe C_0 des contractions T complètement non-unitaires, pour lesquelles il existe une fonction intérieure $u(\lambda)$ (dans le disque unité) [3]) telle que $u(T)=O$; parmi ces fonctions il y a une qui divise les autres; cette fonction, déterminée à coïncidence près, est appelée la fonction minimum de T et est désignée par $m_T(\lambda)$.

Pour $T \in C_0(N)$, la fonction „caractéristique"

$$\Theta_T(\lambda) = [-T + D_{T^*}(I - \lambda T^*)^{-1} D_T] | \mathfrak{D}_T$$

est une fonction analytique à valeurs contractions de $\mathfrak{D}_T$ dans $\mathfrak{D}_{T^*}$, intérieure des deux côtés (c'est-à-dire que $\Theta_T(e^{it})$ est un opérateur unitaire de $\mathfrak{D}_T$ sur $\mathfrak{D}_{T^*}$

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[2]) L'égalité des dimensions des sous-espaces de défaut $\mathfrak{D}_T$ et $\mathfrak{D}_{T^*}$ s'ensuit déjà de ce que $T \in C_{00}$, conséquence de [A], théorème II. 1. 2 et proposition I. 2. 1.

[3]) On désigne par H^∞ l'ensemble des fonctions scalaires $u(\lambda)$, holomorphes et bornées dans le disque $|\lambda| < 1$. La fonction $u \in H^\infty$ est *intérieure* si ses valeurs limites (non-tangentielles) sur le cercle unité sont p.p. de module 1. Pour deux fonctions intérieures, on dit qu'elles *coïncident*, lorsqu'elles ne diffèrent qu'en un facteur constant près (de module 1). Les fonctions intérieures forment, par rapport à la multiplication usuelle, un semi-groupe commutatif, avec l'élément unité $u(\lambda) \equiv 1$. Toute notion arithmétique (multiple, diviseur, etc.) pour les fonctions intérieures sera entendue par rapport à cette structure de semi-groupe.

en presque tous les points du cercle unité). Le déterminant $d_T(\lambda)$ de la matrice de $\Theta_T(\lambda)$, prise par rapport à deux bases orthonormales quelconques dans les espaces de défaut, est une fonction scalaire intérieure telle que $d_T(T)=O$ (l'analogue du théorème de Cayley—Hamilton pour les matrices finies). La fonction minimum m_T s'obtient comme le quotient de d_T par le plus grand diviseur commun intérieur des mineurs d'ordre $N-1$ de la matrice de Θ_T (dans le cas $N=1$ on a $m_T=d_T$); cf. [A], théorème VI. 5. 2.

La restriction T' d'un opérateur $T\in C_0(N)$ à un sous-espace invariant est d'une classe $C_0(N')$ avec $N'\leqq N$ (cf. [A], n° IX. 3).

Notre but principal dans cette Note est de démontrer les deux théorèmes suivants:

Théorème 1. *Pour tout opérateur T de classe $C_0(N)$ dans $\mathfrak{H}$ il existe un sous-espace $\mathfrak{H}_0$ invariant pour T, tel que la restriction $T_0=T|\mathfrak{H}_0$ ait un vecteur cyclique dans $\mathfrak{H}_0$ et que $m_{T_0}=m_T$.*

Théorème 2. *Pour un opérateur $T\in C_0(N)$ dans l'espace $\mathfrak{H}$ les conditions suivantes sont équivalentes:*

(i) *il existe un vecteur cyclique pour T;*

(ii) $m_T=d_T$; [4])

(iii) *pour tout diviseur intérieur u de m_T il existe un sous-espace $\mathfrak{H}_u$ invariant pour T et un seul tel que $m_{T|\mathfrak{H}_u}=u$, il est donné notamment par*

$$\mathfrak{H}_u=\{h\colon h\in\mathfrak{H},\, u(T)h=0\};$$

(iv) *il n'y a pas de sous-espace propre $\mathfrak{L}$ de $\mathfrak{H}$, invariant pour T et tel que $m_{T|\mathfrak{L}}=m_T$;*

(v) *il n'y a pas de sous-espaces différents $\mathfrak{L}_1$ et $\mathfrak{L}_2$, invariants pour T et tels que $T|\mathfrak{L}_1$ et $T|\mathfrak{L}_2$ soient quasi-similaires;*

(vi) *tout opérateur borné X permutant à T est une fonction de T: $X=\varphi(T)$ où $\varphi\in N_T$.* [5])

Ce théorème suggère la définition suivante:

Définition. Les opérateurs $C_0(N)$ (N quelconque) vérifiant les conditions équivalentes (i)—(vi) seront appelés *sans multiplicité.*

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[4]) Pour des fonctions intérieures nous utilisons le signe d'égalité pour indiquer qu'elles coïncident.

[5]) N_T est la classe des fonctions $\varphi=\dfrac{u}{v}$ telles que $u\in H_T^\infty$ et $v\in K_T^\infty$, et on définit:

$$\varphi(T)=v(T)^{-1}u(T);$$

cf. [A], chap. IV (là, on suppose aussi que φ soit holomorphe dans $|\lambda|<1$, mais cette restriction est superflue).

Remarque 1. Le problème de savoir si pour un opérateur $T \in C_0(N)$ dans $\mathfrak{H}$ l'ensemble des vecteurs cycliques est ou bien vide ou bien dense dans $\mathfrak{H}$, est laissé ouvert. (Pour des opérateurs de type général, cet ensemble peut être non-vide *et* non-dense.) [6])

La démonstration des théorèmes 1 et 2 fera l'objet des numéros suivants. Tout d'abord on établira un lemme appartenant à l'arithmétique des fonctions intérieures, dont on fera usage dans la démonstration du théorème 1.

Remarque 2. Si $T \in C_0(N)$ et $\sigma(T) = \{1\}$, alors $m_T = e_a$ avec $a > 0$. Ici on fait usage de la notation

$$e_s(\lambda) = \exp\left(s\frac{\lambda+1}{\lambda-1}\right) \qquad (s \geqq 0);$$

cf. [A], chap. III. Les diviseurs intérieurs de e_a étant les fonctions e_s $(0 \leqq s \leqq a)$, ces diviseurs font un système ordonné par rapport à la divisibilité et par conséquent les sous-espaces

$$\mathfrak{H}_s = \{h\colon e_s(T)h = 0\} \qquad (0 \leqq s \leqq a),$$

invariants pour T, forment un système ordonné par rapport à l'inclusion. Si T est unicellulaire, il n'y a pas d'autres sous-espaces invariants pour T (cf. [A], proposition III. 7. 5), donc T vérifie la condition (iii) du théorème 2, donc T est sans multiplicité. Inversement, si T est sans multiplicité, il n'y a pas en vertu de (iii) d'autres sous-espaces invariants, donc T est unicellulaire. Ainsi, *pour $T \in C_0(N)$ avec $\sigma(T) = \{1\}$, les conditions d'être sans multiplicité et d'être unicellulaire sont équivalentes*. (Cf. [A], théorème IX. 3. 4.)

3. Un lemme sur l'arithmétique des fonctions intérieures

Lemme. *Soient $u_1, \ldots, u_N$ des fonctions intérieures. On peut attacher à chaque u_k des diviseurs intérieurs v_k, v_k' de u_k de sorte qu'on ait*

$$\text{a)} \quad v_k \wedge v_m = 1 \ (k \neq m), \qquad v_1 \vee v_2 \vee \cdots \vee v_N = u_1 \vee u_2 \vee \cdots \vee u_N,$$

et

$$\text{b)} \quad v_k' \wedge v_m' = 1 \ (k \neq m), \qquad v_1' \vee v_2' \vee \cdots \vee v_N' = u_1 \wedge u_2 \wedge \cdots \wedge u_N. \quad ^{7)}$$

Démonstration. Nous envisagerons seulement l'assertion a); l'assertion b) se démontre d'une manière analogue. D'ailleurs, on fera usage dans la suite seulement de l'assertion a).

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[6]) C'est le cas, par exemple, pour la translation unilatérale simple $(x_1, x_2, \ldots) \to (0, x_1, x_2, \ldots)$ dans l'espace complexe l^2.

[7]) On indique par $\wedge$ et $\vee$ le plus grand diviseur intérieur commun, et le plus petit multiple intérieur commun, selon les cas.

Posons, pour abréger, $u_* = u_1 \vee u_2 \vee \cdots \vee u_N$.

1) Supposons d'abord que u_* est un produit de Blaschke; l'ensemble des zéros de u_* dans le disque unité ouvert soit désigné par A. Soit $r_*(a)$ la multiplicité du point $a \in A$ comme zéro de $u_*(\lambda)$, et soit $r_k(a)$ sa multiplicité comme zéro de $u_k(\lambda)$. On a

$$0 \leqq r_k(a) \leqq r_*(a) \quad \text{et} \quad r_*(a) = \max\{r_1(a), \ldots, r_N(a)\}.$$

Définissons, pour $k = 1, \cdots, N$,

$$A_k = \{a\colon a \in A,\ r_*(a) = r_k(a),\ r_*(a) > r_i(a) \text{ pour } i = 1, \ldots, k-1\}.$$

Les ensembles A_k sont disjoints et leur réunion est égale à A.

Soit v_k le produit de Blaschke attaché aux zéros $a \in A_k$, chacun de multiplicité $r_*(a)$. Les propriétés a) découlent d'une manière évidente de ce que $A_k \cap A_m = \emptyset$ $(k \neq m)$ et que $\bigcup_1^N A_k = A$. De plus, comme $v_k(\lambda)$ n'a pas de zéros en dehors de A_k et que les points $a \in A_k$ sont, comme zéros de $v_k(\lambda)$ et $u_k(\lambda)$, de la même multiplicité (égale à $r_*(a)$), on conclut que v_k est un diviseur de u_k.

2) Supposons ensuite que u_* est une fonction intérieure de type

$$u_*(\lambda) = \exp\left[-\int_0^{2\pi} \frac{e^{it}+\lambda}{e^{it}-\lambda}\, d\mu_{*t}\right]$$

où μ_* est une mesure borélienne non-négative finie dans $[0,2\pi)$, singulière par rapport à la mesure de Lebesgue. Les fonctions u_k (diviseurs de u_*) sont alors de même type; la mesure attachée à u_k soit μ_k. Comme μ_k est majorée par μ_*, on a

$$\mu_k(\omega) = \int_\omega r_k(t)\, d\mu_{*t} \qquad (k = 1, \ldots, N;\quad \omega \text{ borélien dans } [0,2\pi)),$$

les fonctions $r_k(t)$ étant boréliennes, définies p.p. par rapport à μ_*, et telles que $0 \leqq r_k(t) \leqq 1$; de plus on a p.p. par rapport à μ_*

$$\max\{r_1(t), \ldots, r_N(t)\} = 1.$$

Soit

$$A_k = \{t\colon t \in [0,2\pi),\ r_k(t) = 1,\ r_i(t) < 1 \text{ pour } i = 1, \ldots, k-1\}.$$

Les ensembles $A_1, \cdots, A_N$ fournissent une décomposition de $[0,2\pi)$ en des parties boréliennes disjointes (modulo des ensembles de μ_*-mesure nulle).

Définissons:

$$v_k(\lambda) = \exp\left[-\int_{A_k} \frac{e^{it}+\lambda}{e^{it}-\lambda}\, d\mu_{*t}\right] \qquad (k = 1, \ldots, N).$$

Les relations a) en résultent aussitôt. De plus v_k est un diviseur de u_k. Cela s'ensuit de ce que, pour ω borélien quelconque,

$$\mu_*(\omega \cap A_k) = \int_{\omega \cap A_k} d\mu_{*t} = \int_{\omega \cap A_k} r_k(t)\, d\mu_{*t} \leqq \int_{\omega} r_k(t)\, d\mu_{*t} = \mu_k(\omega).$$

3) Le cas général résulte des cas particuliers précédents en prenant les factorisations $u_k = u_k^{(1)} u_k^{(2)}$ où $u_k^{(1)}$ est le facteur de Blaschke et $u_k^{(2)}$ est le facteur de type „singulier" de u_k. En construisant les fonctions $v_k^{(1)}$ attachées aux $u_k^{(1)}$ d'après 1), et les fonctions $v_k^{(2)}$ attachées aux $u_k^{(2)}$ d'après 2), les produits

$$v_k = v_k^{(1)} \cdot v_k^{(2)} \qquad (k = 1, \ldots, N)$$

fournissent un système de type exigé.

4. Démonstration du théorème 1

Comme $T^{*n} \to O$ $(n \to \infty)$, on a

$$h = \sum_{0}^{\infty} T^n (I - TT^*) T^{*n} h \quad \text{pour tout} \quad h \in \mathfrak{H},$$

d'où il s'ensuit que si l'on choisit dans le sous-espace de défaut $\mathfrak{D}_{T^*}$ (de dimension N) une base quelconque $f_1, \cdots, f_N$, on aura

$$\mathfrak{H} = \bigvee_{i=1}^{N} \mathfrak{L}_i \quad \text{où} \quad \mathfrak{L}_i = \bigvee_{n=0}^{\infty} T^n f_i, \tag{4.1}$$

les sous-espaces $\mathfrak{L}_i$ étant invariants pour T. Posons $T_i = T|\mathfrak{L}_i$. D'après [A], proposition III. 6. 2 (qui se généralise du cas de deux sous-espaces au cas d'un nombre quelconque de sous-espaces) on a pour les fonctions minimum correspondantes:

$$m_T = m_{T_1} \vee m_{T_2} \vee \cdots \vee m_{T_N}.$$

Posons $u_i = m_{T_i}$ $(i = 1, \cdots, N)$ et choisissons pour chaque u_i un diviseur intérieur v_i selon le lemme, et soit $w_i = u_i / v_i$. Posons

$$f_i' = w_i(T) f_i \quad (i = 1, \ldots, N) \quad \text{et} \quad \mathfrak{L}_i' = \bigvee_{n=0}^{\infty} T^n f_i'.$$

On a alors évidemment

$$\mathfrak{L}_i' = \overline{w_i(T)\mathfrak{L}_i} = \overline{w_i(T_i)\mathfrak{L}_i},$$

d'où il s'ensuit que $\mathfrak{L}_i'$ est invariant pour T_i. De plus on a

$$v_i(T_i)\mathfrak{L}_i' \subset \overline{v_i(T_i) w_i(T_i) \mathfrak{L}_i} = \overline{u_i(T_i)\mathfrak{L}_i} = \{0\} \quad \text{puisque} \quad u_i = m_{T_i},$$

donc en posant $T_i' = T_i|\mathfrak{L}_i'$ $(= T|\mathfrak{L}_i')$ on a $v_i(T_i') = O$ et par conséquent v_i est divisible par $m_{T_i'}$:

$$v_i = m_{T_i'} \cdot p_i \quad \text{(avec } p_i \text{ intérieure).}$$

Soit

$$T_i = \begin{bmatrix} T_i' & * \\ O & T_i'' \end{bmatrix}$$

la triangulation de T_i correspondant à la décomposition $\mathfrak{L}_i = \mathfrak{L}_i' \oplus \mathfrak{L}_i''$. Comme $\mathfrak{L}_i' = \overline{w_i(T_i)\mathfrak{L}_i}$, on a

$$\mathfrak{L}_i'' = \{h : h \in \mathfrak{L}_i, w_i(T_i)^* h = 0\} = \{h : h \in \mathfrak{L}_i, w_i^{\sim}(T_i^*) h = 0\}.^{8)}$$

Cela entraîne que $T_i''^*$ $(= T_i^*|\mathfrak{L}_i'')$ a sa fonction minimum égale à $w_i^{\sim}$. Donc

$$m_{T_i''} = w_i.$$

Or on sait que m_{T_i} est un diviseur de $m_{T_i'} \cdot m_{T_i''}$ (*cf.* [A], proposition III. 6. 1); par conséquent $u_i p_i$ $(= m_{T_i} p_i)$ est un diviseur de u_i $(= v_i w_i = m_{T_i'} p_i \cdot m_{T_i''})$. Cela entraîne $p_i = 1$ $(i = 1, \cdots, N)$, donc

$$m_{T_i'} = v_i.$$

Posons

$$g = \sum_1^N f_i', \quad \mathfrak{H}_0 = \bigvee_{n=0}^{\infty} T^n g, \quad T_0 = T|\mathfrak{H}_0 \quad \text{et} \quad p = m_{T_0}.$$

On a alors

$$\sum_{i=1}^N p(T) f_i' = p(T) g = p(T_0) g = 0,$$

d'où

$$\mathfrak{L}_j' \ni p(T) f_j' = -\sum_{i \neq j} p(T) f_i' \in \bigvee_{i \neq j} \mathfrak{L}_i',$$

donc

$$p(T) f_j' \in \mathfrak{M}_j \quad \text{où} \quad \mathfrak{M}_j = \mathfrak{L}_j' \cap (\bigvee_{i \neq j} \mathfrak{L}_i').$$

$\mathfrak{M}_j$ est invariant pour T et la fonction minimum q_j de $T|\mathfrak{M}_j$ est un diviseur commun de v_j et de $\bigvee_{i \neq j} v_i$; puisque $v_j \wedge v_i = 1$ pour $i \neq j$, cela entraîne $q_j = 1$, d'où $\mathfrak{M}_j = \{0\}$ et par conséquent $p(T) f_j' = 0$. Comme

$$\mathfrak{L}_j' = \bigvee_{n=0}^{\infty} T^n f_j',$$

il en dérive que $p(T)\mathfrak{L}_j' = \{0\}$, donc $p(T_j') = O$. Ainsi, p est divisible par $v_j = m_{T_j'}$

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[8]) On se sert de la notation $u^{\sim}(\lambda) = \overline{u(\bar{\lambda})}$.

$(j=1, \cdots, N)$ et par conséquent aussi par $v_1 \vee v_2 \vee \cdots \vee v_N = u_1 \vee u_2 \vee \cdots \vee u_N = m_T$. D'autre part, $p\,(=m_{T_0})$ est un diviseur de m_T. On conclut que

$$p=m_T, \quad m_{T_0}=m_T.$$

Le sous-espace $\mathfrak{H}_0$ jouit donc des propriétés énoncés dans le théorème 1: $T_0=T|\mathfrak{H}_0$ admet le vecteur cyclique g dans $\mathfrak{H}_0$ et les fonctions minimum de T_0 et T coïncident.

Remarque. Il convient d'observer que cette démonstration porte non seulement pour les opérateurs de classe $C_0(N)$, mais pour toute contraction T dans $\mathfrak{H}$ pour laquelle il existe un nombre fini d'éléments $f_i\,(i=1, \cdots, N)$ de sorte que (1. 1) soit vérifié, ou, en d'autres termes, que $\dim_T \mathfrak{H}$ soit finie (cf. [B]).

5. Démonstration du théorème 2: première partie

Dans cette première partie de la démonstration on établira, pour $T\in C_0(N)$, l'équivalence des conditions (i)—(iv) et cela en démontrant les implications (i) → (ii) → (iii) → (iv) → (i).

(i) → (ii): *cf.* [A], proposition IX. 3. 3.

(ii) → (iii): Soit $\mathfrak{L}$ un sous-espace invariant pour T et posons $u=m_{T|\mathfrak{L}}$. Il est manifeste que

$$\mathfrak{L} \subset \mathfrak{H}_u = \{h\colon h\in\mathfrak{H}, u(T)h = 0\}.$$

Posons $\mathfrak{M}=\mathfrak{H}_u \ominus \mathfrak{L}$ et considérons la triangulation de $T_u=T|\mathfrak{H}_u$ correspondant à la décomposition $\mathfrak{H}_u=\mathfrak{L}\oplus\mathfrak{M}$, soit

$$T_u = \begin{bmatrix} A & * \\ O & B \end{bmatrix}.$$

On a alors

$$d_{T_u} = d_A \cdot d_B$$

par [A], lemme IX. 3. 1.

Comme c'était montré au cours de la démonstration de la proposition IX. 3. 2 de [A], la condition $m_T=d_T$ entraîne $m_{T_1}=d_{T_1}$ pour toute restriction T_1 de T à un sous-espace invariant. On a donc en particulier

$$d_{T_u}=m_{T_u} \quad \text{et} \quad d_A=d_{T|\mathfrak{L}}=m_{T|\mathfrak{L}}.$$

Or $m_{T|\mathfrak{L}}=u$ par la définition de u et $m_{T_u}=u$ en vertu de [A], théorème III. 6. 3.

Il en résulte que

$$u = u \cdot d_B,$$

donc $d_B=1$, ce qui entraîne $\mathfrak{M}=\{0\}$, $\mathfrak{L}=\mathfrak{H}_u$.

(iii)$\to$(iv): implication évidente.

(iv)$\to$(i). Soit $\mathfrak{H}_0$ un sous-espace invariant pour T tel que $T_0 = T|\mathfrak{H}_0$ admet un vecteur cyclique g dans $\mathfrak{H}_0$ et que $m_{T_0} = m_T$; l'existence de tel $\mathfrak{H}_0$ est affirmée par le théorème 1. Or par (iv) l'égalité $m_{T_0} = m_T$ entraîne $\mathfrak{H}_0 = \mathfrak{H}$. Donc g est cyclique pour T dans $\mathfrak{H}$.

6. Quelques conséquences

Avant de compléter la démonstration du théorème 2, nous indiquons quelques conséquences de ce qui a été déjà prouvé.

Proposition 1. *Si un opérateur T de classe $C_0(N)$ admet un vecteur cyclique, il en est de même de T^* ainsi que des restrictions de T à des sous-espaces invariants pour T.*

Démonstration. Pour $T \in C_0(N)$ on a aussi $T^* \in C_0(N)$, avec

$$d_{T^*} = d_{\tilde{T}} \quad \text{et} \quad m_{T^*} = m_{\tilde{T}}.$$

La condition (ii) du théorème 1 se transfère donc de T à T^*. D'autre part, la condition (iii) pour T entraîne évidemment la même condition pour la restriction de T à un sous-espace invariant. Ainsi, la proposition résulte de l'équivalence des conditions (i), (ii) et (iii).

Proposition 2. *Tout opérateur $T \in C_0(N)$ admettant un vecteur cyclique est quasi-similaire à un opérateur $S \in C_0(1)$.*

Démonstration. Comme T et (en vertu de la proposition 1) T^* admettent des vecteurs cycliques, il s'ensuit par la proposition 1 de la Note [B] que T et T^* ont des transformées quasi-affines S et S_* de classe $C_0(1)$. T est alors une transformée quasi-affine de $Z = S_*^*$ et par conséquent S est une transformée quasi-affine de Z. Puisque Z appartient à $C_0(1)$ ensemble avec S_*, on a deux opérateurs de classe $C_0(1)$, S et Z, dont S est une transformée quasi-affine de Z, donc, en vertu de [A], Corollaire VI. 5. 3, S et Z sont unitairement équivalents. On conclut que T est quasi-similaire à S.

Proposition 3. *Pour que deux opérateurs $T_i \in C_0(N_i)$ $(i = 1, 2)$, admettant des vecteurs cycliques, soient quasi-similaires, il est nécessaire et suffisant que leurs fonctions minimum coïncident.*

Démonstration. La nécessité de la condition $m_{T_1} = m_{T_2}$ résulte comme un cas particulier de [A], proposition III. 4. 6. La suffisance de la condition se démontre comme il suit. Par la proposition 2 ci-dessus il existe des opérateurs S_i $(i = 1, 2)$ de classe $C_0(1)$ tels que S_i est quasi-similaire à T_i. Comme $m_{S_1} = m_{T_1} =$

$=m_{T_2}=m_{S_2}$, il s'ensuit de [A], corollaire VI. 5. 3, que S_1 et S_2 sont unitairement équivalents. Il en résulte que T_1 et T_2 sont quasi-similaires au même opérateur S_1, donc aussi quasi-similaires l'un à l'autre.

Proposition 4. *Tout sous-espace invariant $\mathfrak{H}_0$ vérifiant le théorème 1 est maximal dans le sens que si $\mathfrak{H}_1$ est un sous-espace invariant pour T, comprenant $\mathfrak{H}_0$ et tel que $T_1=T|\mathfrak{H}_1$ admet un vecteur cyclique, on a $\mathfrak{H}_1=\mathfrak{H}_0$.*

Démonstration. Comme $T_0=T|\mathfrak{H}_0$ peut être considéré comme une restriction de T_1 qui, à son tour, est une restriction de T, m_{T_0} est un diviseur de m_{T_1} et m_{T_1} est un diviseur de m_T. Puisque $m_{T_0}=m_T$, cela entraîne $m_{T_0}=m_{T_1}$. En vertu de l'équivalence des conditions (i) et (iv) (appliquée à T_1) on a donc $\mathfrak{H}_0=\mathfrak{H}$.

7. Démonstration du théorème 2 : conclusion

Pour achever la démonstration du théorème 2 il suffit d'établir les implications suivantes: (iii) $\to$ (v) $\to$ (i) $\to$ (vi) $\to$ (i).

(iii) $\to$ (v): C'est évident puisque les fonctions minimum de deux contractions quasi-similaires, de classe C_0, coïncident; cf. [A], proposition III. 4. 6.

(v) $\to$ (i): Soit $\mathfrak{H}_0$ le sous-espace invariant pour T, vérifiant le théorème 1, donc tel que $T_0=T|\mathfrak{H}_0$ admet un vecteur cyclique et que $m_{T_0}=m_T$. Il s'agit de montrer que $\mathfrak{H}_0=\mathfrak{H}$.

Or, dans le cas contraire il existe un $h\in\mathfrak{H}\ominus\mathfrak{H}_0$, $h\neq 0$. Soit $\mathfrak{L}_1$ le sous-espace invariant pour T, engendré par h, et soit $T_1=T|\mathfrak{L}_1$; T_1 admet alors le vecteur cyclique h et m_{T_1} est un diviseur de m_T, donc de m_{T_0}. Par conséquent il existe un sous-espace $\mathfrak{L}_2$ de $\mathfrak{H}_0$, invariant pour T_0 et tel que, en posant $T_2=T_0|\mathfrak{L}_2$ $(=T|\mathfrak{L}_2)$, on a $m_{T_2}=m_{T_1}$. Comme T_0 admet un vecteur cyclique, il en est de même de T_2 en vertu de la proposition 1 du n° précédent. Ainsi, T_1 et T_2 admettent des vecteurs cycliques et $m_{T_1}=m_{T_2}$. Comme $T\in C_0(N)$ entraîne aussi $T_i\in C_0(N_i)$ $(0\leqq N_i\leqq N)$, il s'ensuit de la proposition 3 du numéro précédent que T_1 et T_2 sont quasi-similaires. Comme, évidemment, $\mathfrak{L}_1\neq\mathfrak{L}_2$, nous sommes aboutis à une contradiction avec (v). Ainsi, $\mathfrak{H}_0=\mathfrak{H}$.

(i) $\to$ (vi): En vertu de la proposition 2 du numéro précédent, T est quasi-similaire à un opérateur $S\in C_0(1)$, donc il y a des quasi-affinités A, B telles que

$$AS=TA,\quad BT=SB. \tag{7.1}$$

Soit X un opérateur borné quelconque tel que $XT=TX$. Grâce à (7.1) on a alors

$$SBXA=BXAS,$$

donc il résulte par le théorème de SARASON [S] (*cf.* aussi [C]) que BXA est une fonction de S:

(7. 2) $$BXA = u_X(S) \quad \text{où} \quad u_X \in H^\infty.$$

Comme $AS = TA$ entraîne $Au(S) = u(T)A$ pour $u \in H^\infty$ quelconque, on déduit de (7. 2) que

$$ABXA = Au_X(S) = u_X(T)A,$$

d'où, vu que le domaine de A est dense,

(7. 3) $$ABX = u_X(T).$$

Dans le cas particulier $X = I$ cela donne

(7. 4) $$AB = u_I(T).$$

Comme AB est une quasi-affinité, $u_I(T)$ admet un inverse à domaine dense, donc u_I appartient à la classe des fonctions K_T^∞ (cf. [A], n° III. 3. 2). Comme de plus $u_X \in H^\infty = H_T^\infty$, il résulte de (7. 3) et (7. 4) que

$$X = \varphi(T) \quad \text{où} \quad \varphi = u_X/u_I \in N_T.$$

(vi) $\to$ (i): D'après le theorème 1 il existe un sous-espace $\mathfrak{H}_0$ tel que $T_0 = T|\mathfrak{H}_0$ admet un vecteur cyclique et que $m_{T_0} = m_T$. Il s'agit de montrer que (vi) entraîne $\mathfrak{H}_0 = \mathfrak{H}$.

Supposons le contraire, c'est-à-dire que $\mathfrak{H}' = \mathfrak{H} \ominus \mathfrak{H}_0 \neq \{0\}$. D'après la proposition 2, T_0 est quasi-similaire à un opérateur $S \in C_0(1)$ dans un espace $\mathfrak{G}$. Donc il existe une quasi-affinité A_0 de $\mathfrak{G}$ dans $\mathfrak{H}_0$ telle que $T_0A_0 = A_0S$. En regardant A_0 comme un opérateur A de $\mathfrak{G}$ dans $\mathfrak{H}$, on a donc $TA = AS$.

Appliquons maintenant le théorème 1 et la proposition 2 à l'opérateur $T' = T^*|\mathfrak{H}'$. Il résulte qu'il existe un sous-espace $\mathfrak{H}_0'$ de $\mathfrak{H}'$, invariant pour T' et tel que, en posant $T_0' = T'|\mathfrak{H}_0'$, on ait $m_{T_0'} = m_{T'}$ et que T_0' soit quasi-similaire à un opérateur $S' \in C_0(1)$ dans un espace $\mathfrak{G}'$. Donc il existe une quasi-affinité B_0 de $\mathfrak{G}'$ dans $\mathfrak{H}_0'$ telle que $T_0'B_0 = B_0S'$. Comme $\mathfrak{H}_0' \subset \mathfrak{H}$, on peut regarder B_0 aussi comme un opérateur B de $\mathfrak{G}'$ dans $\mathfrak{H}$; puisque $T_0' \subset T' \subset T^*$, on aura $T^*B = BS'$. En prenant les adjoints on obtient $B^*T = ZB^*$ où B^* est un opérateur de $\mathfrak{H}$ dans $\mathfrak{G}'$ et $Z = S'^*$; Z est un opérateur dans $\mathfrak{G}'$, de classe $C_0(1)$ et tel que $m_Z = m_{\tilde{S}'} = m_{\tilde{T}_0'}$, donc m_Z est un diviseur de $m_{\tilde{T}^*}$, c'est-à-dire de m_T. Comme de plus $m_T = m_{T_0} = m_S$, il résulte que m_Z est un diviseur de m_S. Ainsi, il existe une restriction de S à un sous-espace invariant dont la fonction minimum est égale à m_Z, par conséquent (puisqu'il s'agit d'opérateurs de classe $C_0(1)$) cette restriction est unitairement équivalente à Z (cf. [A], remarque à la fin du n° VI. 5. 1). Donc on peut supposer que Z est la restriction de S à un sous-espace $\mathfrak{G}'$ de $\mathfrak{G}$, invariant pour S.

Pour $h_0 \in \mathfrak{H}_0$ et $g' \in \mathfrak{G}'$ on a $(B^* h_0, g') = (h_0, Bg') = (h_0, B_0 g') = 0$ parce que $B_0 g' \in \mathfrak{H}_0' \subset \mathfrak{H}' \perp \mathfrak{H}_0$. Ainsi, $B^* h_0 = 0$, donc $B^* \mathfrak{H}_0 = \{0\}$.

Posons $X = AB^*$. C'est un opérateur borné dans $\mathfrak{H}$, permutant à T. En effet,

$$TAB^* = ASB^* = AZB^* = AB^* T.$$

En vertu de (vi) on a donc $X = \varphi(T)$ avec $\varphi = \frac{u}{v} \in N_T$, donc $v(T)X = u(T)$. Cela entraîne

$$u(T_0) = u(T)|\mathfrak{H}_0 = v(T)AB^*|\mathfrak{H}_0 = O,$$

d'où il s'ensuit que u est un multiple de m_{T_0}, donc de m_T. Par conséquent, on a $u(T) = O$, d'où $X = v(T)^{-1} u(T) = O$. Or, comme A est inversible, $AB^* = X = O$ entraîne $B^* = O$, donc $B = O$, $B_0 = O$, ce qui contredit le fait que B_0 est une quasi-affinité de l'espace $\mathfrak{G}' \neq \{0\}$ (dans $\mathfrak{H}_0'$).

Cette contradiction démontre que $\mathfrak{H}_0 = \mathfrak{H}$.

8. Opérateurs quasi-similaires, mais non similaires

1. Les propositions 2 et 3 du numéro 6 établissent la quasi-similitude entre certains opérateurs $T_i \in C_0(N_i)$ $(i = 1, 2)$. Vu les analogies entre les opérateurs des classes $C_0(N)$ et les opérateurs dans les espaces de dimension finie, on peut se demander si cette quasi-similitude est même une similitude? On montrera que cela n'est pas le cas. Notamment, on donnera un exemple d'un opérateur T de classe $C_0(2)$, qui est quasi-similaire, mais non similaire, à un opérateur de classe $C_0(1)$.

Nous commençons par démontrer un fait d'un certain intérêt en soi-même.

Proposition 5. *Soit $T \in C_0(N)$ et soit $\Omega_T(\lambda) = [\omega_{ik}(\lambda)]$ $(i, k = 1, \dots, N)$ l'adjointe algébrique de la matrice de la fonction caractéristique $\Theta_T(\lambda)$. Pour que T soit similaire à un opérateur $S \in C_0(1)$ il est nécessaire qu'il existe des fonctions $x_k(\lambda) \in H^\infty$ telles que*

$$\sum_{i=1}^{N} \sum_{k=1}^{N} \omega_{ik} x_{ik} = 1. \tag{8.1}$$

Démonstration. Supposons que T est dans l'espace $\mathfrak{H}$, S est dans l'espace $\mathfrak{H}'$, et qu'il existe des opérateurs bornés $X: \mathfrak{H} \to \mathfrak{H}'$ et $Y: \mathfrak{H}' \to \mathfrak{H}$ tels que

$$SX = XT, \quad TY = YS \quad \text{et} \quad Y = X^{-1}.$$

Comme S est sans multiplicité $(m_S = d_S = \Theta_S)$, il en est de même de T (donc $m_T = d_T$), de plus on a $m_S = m_T$. On peut représenter T et S par leurs modèles fonctionnels dans les espaces

$$\mathfrak{H} = H^2(E^N) \ominus \Theta_T H^2(E^N) \quad \text{et} \quad \mathfrak{H}' = H^2 \ominus m_T H^2, \tag{8.2}$$

ce qui entraîne pour X et Y les représentations

$$X = P_{\mathfrak{H}'}\Phi|\mathfrak{H} \quad \text{et} \quad Y = P_{\mathfrak{H}}\Psi|\mathfrak{H}' \tag{8.3}$$

moyennant deux fonctions analytiques bornées,

$$\Phi \in H^{\infty}(E^N, E^1) \quad \text{et} \quad \Psi \in H^{\infty}(E^1, E^N), \tag{8.4}$$

telles que

$$\Phi\Theta_T H^2(E^N) \subset \Theta_S H^2 = m_T H^2 \quad \text{et} \quad \Psi\Theta_S H^2 \subset \Theta_T H^2(E^N); \tag{8.5}$$

cf. [C]. Vu que $XY = I_{\mathfrak{H}'}$, (8. 3) entraîne

$$I_{\mathfrak{H}'} = P_{\mathfrak{H}'}\Phi P_{\mathfrak{H}}\Psi|\mathfrak{H}'; \tag{8.6}$$

comme de plus, par (8. 2) et (8. 5),

$$P_{\mathfrak{H}'}\Phi(I - P_{\mathfrak{H}})\Psi\mathfrak{H}' \subset P_{\mathfrak{H}'}\Phi\Theta_T H^2(E^N) \subset P_{\mathfrak{H}'} m_T H^2 = \{0\},$$

on a aussi

$$I_{\mathfrak{H}'} = P_{\mathfrak{H}'}\Phi\Psi|\mathfrak{H}', \tag{8.7}$$

donc, pour tout $h' \in \mathfrak{H}'$, $\Phi\Psi h' - h' \in m_T H^2$. En choisissant en particulier $h' = 1 - \overline{m_T(0)} m_T$, on conclut qu'il existe une fonction $u \in H^2$ telle que

$$\Phi\Psi - 1 = m_T u. \tag{8.8}$$

Pour presque tous les points z du cercle unité

$$|u(z)| = |m_T(z)|\,|u(z)| = |\Phi(z)\Psi(z) - 1| \leqq \|\Phi\|_\infty \|\Psi\|_\infty + 1,$$

d'où $u \in H^\infty$.

En vertu de la première des relations (8. 5) on peut attacher à tout $u^N \in H^2(E^N)$ un $v \in H^2$ de la sorte qu'on ait $\Phi\Theta_T u^N = m_T v$; puisque m_T est une fonction intérieure, on a $\|v\|_{H^2} = \|m_T v\|_{H^2} = \|\Phi\Theta_T u^N\|_{H^2} \leqq \|\Phi\|_\infty \|u^N\|_{H^2(E^N)}$, donc $v = \Phi_1 u^N$ définit un opérateur (linéaire) borné $\Phi_1 \colon H^2(E^N) \to H^2$. Il est manifeste que Φ_1 permute à la multiplication par z, d'où il s'ensuit qu'il existe une fonction $\Phi_1(\lambda) \in H^\infty(E^N, E^1)$ de la sorte que $(\Phi_1 u^N)(\lambda) = \Phi_1(\lambda) u^N(\lambda)$. Donc on a

$$\Phi(\lambda)\Theta_T(\lambda) = m_T(\lambda)\Phi_1(\lambda),$$

d'où

$$m_T\Phi = \Phi m_T = \Phi\Theta_T\Omega_T = m_T\Phi_1\Omega_T$$

et par conséquent

$$\Phi = \Phi_1\Omega_T. \tag{8.9}$$

En combinant (8. 9) avec (8. 8) et en observant que $m_T = d_T = (\Omega_T\Theta_T)_{11}$, on obtient

$$\Phi_1\Omega_T\psi - (\Omega_T\Theta_T)_{11} u = 1.$$

Or, les fonctions (scalaires ou opératorielles) u, Θ_T, Φ_1 et Ψ étant analytiques et bornées dans le disque unité, il en résulte l'existence des fonctions $x_{ik}\in H^\infty$, vérifiant (8. 2).

Cela achève la démonstration de la proposition 5.

2. Envisageons maintenant les fonctions intérieures

$$u(\lambda)=\exp\left(\frac{\lambda+1}{\lambda-1}\right) \quad \text{et} \quad v(\lambda)=\prod_{k=1}^{\infty}\frac{a_k-\lambda}{1-a_k\lambda}$$

où $a_k=1-1/k^2$; u et v sont premières entre elles. La fonction matricielle

$$\Theta(\lambda)=\begin{bmatrix} u(\lambda)/\sqrt{2} & u(\lambda)/\sqrt{2} \\ v(\lambda)/\sqrt{2} & -v(\lambda)/\sqrt{2} \end{bmatrix}$$

est intérieure des deux côtés, et contractive pure. Donc il existe un opérateur $T\in C_0(2)$ dont la fonction caractéristique coïncide avec $\Theta(\lambda)$. Comme l'adjointe algébrique de $\Theta(\lambda)$ est la matrice

$$\Theta^A(\lambda)=\begin{bmatrix} -v(\lambda)/\sqrt{2} & -v(\lambda)/\sqrt{2} \\ -u(\lambda)/\sqrt{2} & u(\lambda)/\sqrt{2} \end{bmatrix},$$

dont les éléments sont premiers entre eux, on a $m_T(\lambda)=d_T(\lambda)$, donc T est sans multiplicité et par conséquent quasi-similaire à l'opérateur $S\in C_0(1)$ pour lequel $m_S=m_T$. Mais T n'est pas similaire à S puisque la relation

$$u(\lambda)x(\lambda)+v(\lambda)y(\lambda)=1 \qquad (x, y\in H^\infty)$$

est impossible; en effet, $v(a_k)=0$ pour $k=1, 2, \cdots$ et $u(a_k)\to 0$ lorsque $k\to\infty$.

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On the operator equation $S^*XT=X$ and related topics

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1. If U_+ is the unilateral shift of multiplicity one, then BROWN and HALMOS showed in [1] that the identity $U_+^*XU_+=X$ characterizes the class of Toeplitz operators. In this paper we determine the class of solutions to $S^*XT=X$ for arbitrary contractions S and T on Hilbert space. We show first in § 2 that we can reduce to the case of isometries and then in § 3 we determine the solutions for such. The form the latter solution takes is the same as for the case of the unilateral shift, namely, the class of solutions consists of the compressions of the intertwining operators between their unitary extensions. In § 4 we investigate when intertwining maps exist between unitary operators. In § 5 we investigate the inequalities $T^*XT\geqq X$ and $T^*XT\leqq X$ for a contraction T and Hermitian operators X. We show first that we can reduce a solution of either to a "pure" positive solution of the latter. These we study with the aid of a construction of SZ.-NAGY and FOIAŞ [9] and a recent result they proved on the intertwining maps for contractions [10]. As corollaries we obtain results analogous to those obtained in §§ 2 and 3. We also obtain a result due to PUTNAM [8] and certain facts about hyponormal operators.

In § 6 we investigate these same equations in the presence of various hypotheses of compactness. As corollaries we obtain a lemma of DYE [2], a generalization of the result that the only compact Toeplitz operator is 0, and a further proof of the result that a compact hyponormal operator is normal. In the last section we briefly explore the form our results take when T is identified as the Cayley transform of an accretive operator.

2. We make use of some of the more elementary aspects of the theory for contractions due to SZ.-NAGY and FOIAŞ [9] and begin by introducing a few of their ideas.

Let T be a contraction on the complex Hilbert space $\mathfrak{H}$. From the inequality $T^{*n}(I-T^*T)T^n\geqq 0$ it follows that the sequence $\{T^{*n}T^n\}$ is monotonically decreasing and hence converges strongly to a positive contraction. If we denote the unique positive square root of this contraction by A_T, then A_T is 0 if and only if the sequence $\{T^n\}$ converges to 0 in the strong operator topology. Moreover, since $T^*A_T^2T=A_T^2$ we see that A_T^2 is a solution to the equation $T^*XT=X$.

Let $\mathfrak{M}_T$ denote the closure of the range of A_T and define V_T by $V_T A_T x = A_T T x$ on the range of A_T. From the identity

$$\|V_T A_T x\|^2 = \|A_T T x\|^2 = (T^* A_T^2 T x, x) = (A_T^2 x, x) = \|A_T x\|^2$$

it follows that V_T is well defined and can be uniquely extended to an isometry on $\mathfrak{M}_T$ which we also denote by V_T. If by abuse of language we allow A_T to denote operators from $\mathfrak{M}_T$ to $\mathfrak{H}$ and from $\mathfrak{H}$ to $\mathfrak{M}_T$ as well as an operatoı from $\mathfrak{H}$ to $\mathfrak{H}$, then the identities $V_T A_T = A_T T$ and $A_T V_T^* = T^* A_T$ can be seen to hold. This convention will be extremely useful and should cause no confusion.

If S is a contraction on $\mathfrak{H}$ and T is a contraction on $\mathfrak{K}$, we denote by $\mathfrak{S}(S, T)$ the collection of operators X in $\mathfrak{L}(\mathfrak{K}, \mathfrak{H})$, the space of bounded operators from $\mathfrak{K}$ to $\mathfrak{H}$, satisfying the equation $S^*XT = X$. It is easy to verify that $\mathfrak{S}(S, T)$ is a subspace of $\mathfrak{L}(\mathfrak{K}, \mathfrak{H})$, which is closed in the weak operator topology. In the special case $S = T$, the subspace $\mathfrak{S}(T, T)$ $(= \mathfrak{S}_T)$ is closed under the adjoint operation so that it is spanned by its Hermitian elements. (We shall see that it is also spanned by its positive elements.)

In the following theorem we show how to reduce the solution of the equation $S^*XT = X$ to that of $V_S^* Y V_T = Y$.

Theorem 1. *Let S be a contraction on $\mathfrak{H}$ and T be a contraction on $\mathfrak{K}$. Then $\mathfrak{S}(S, T) = A_S \mathfrak{S}(V_S, V_T) A_T$. Moreover, every X in $\mathfrak{S}(S, T)$ can be represented in the form $X = A_S Y A_T$ with Y in $\mathfrak{S}(V_S, V_T)$ such that $\|Y\| = \|X\|$.*

Proof. Let X be a contraction in $\mathfrak{L}(\mathfrak{K}, \mathfrak{H})$ so that $S^*XT = X$. Then $S^*XX^*S \geqq$ $\geqq S^*XTT^*X^*S = XX^*$ so that by induction we obtain $S^{*n}XX^*S^n \geqq XX^*$ for all n. Thus $S^{*n}S^n \geqq XX^*$ for all n and from the definition of A_S it follows that $A_S^2 \geqq XX^*$. Hence there exists a contraction C_0 from $\mathfrak{M}_S$ to $\mathfrak{H}$ such that $X^* = C_0 A_S$ and, taking adjoints, a contraction C $(= C_0^*)$ from $\mathfrak{H}$ to $\mathfrak{M}_S$ so that $X = A_S C$. Substituting this in our equation we obtain $A_S V_S^* C T = S^* A_S C T = A_S C$. Since the range of both V_S^* and C is contained in $\mathfrak{M}_S$ and A_S is one-to-one on $\mathfrak{M}_S$, we obtain $V_S^* C T = C$. Repeating our previous argument we have $T^*C^*CT \geqq T^*C^*V_S V_S^* CT = C^*C$ from which it follows as before that $A_T^2 \geqq C^*C$. Hence there exists a contraction Y from $\mathfrak{M}_T$ to $\mathfrak{H}$ so that $C = YA_T$. Substituting we have $V_S^* Y V_T A_T = V_S^* Y A_T T = V_S^* C T =$ $= C = YA_T$, and hence $V_S^* Y V_T = Y$. Thus, the operatoı X can be written $X = A_S Y A_T$, with Y in $\mathfrak{S}(V_S, V_T)$, so we have shown $\mathfrak{S}(S, T)$ is contained in $A_S \mathfrak{S}(V_S, V_T) A_T$.

To prove the converse suppose that Y is in $\mathfrak{S}(V_S, V_T)$. Then we find that $S^* A_S Y A_T T = A_S V_S^* Y V_T A_T = A_S Y A_T$ so that $X = A_S Y A_T$ is in $\mathfrak{S}(S, T)$. Moreover since we have shown that if X is a contraction, then Y can be taken to be a contraction, we have then for a general X in $\mathfrak{S}(S, T)$ that Y can be represented in the form $X = A_S Y A_T$ with Y in $\mathfrak{S}(V_S, V_T)$ and such that $\|Y\| = \|X\|$. This completes the proof that $\mathfrak{S}(S,T)$ is equal to $A_S \mathfrak{S}(V_S, V_T) A_T$.

From this it follows that if either A_T or A_S is 0, then $\mathfrak{S}(S, T)=(0)$. The question of necessary and sufficient conditions for $\mathfrak{S}(S, T)\neq(0)$ must wait for a detailed study of the case of isometries. We can at this point determine the situation in case $S=T$.

Corollary 2. 1. *Let T be a contraction on $\mathfrak{H}$. Then $\mathfrak{S}(T, T)$ is (0) if and only if $A_T=0$.*

Proof. From the theorem we have that $A_T=0$ implies $\mathfrak{S}(T, T)=(0)$. If $A_T\neq 0$, then since $0\neq A_T^2$ is in $\mathfrak{S}(T, T)$ it follows that $\mathfrak{S}(T, T)\neq(0)$.

3. For a Hilbert space $\mathfrak{D}$ we let $\boldsymbol{H}_{\mathfrak{D}}$ denote the space of functions f from the non negative integers $\boldsymbol{Z}^+$ to $\mathfrak{D}$ so that $\sum_{n=0}^{\infty}\|f(n)\|^2<\infty$. The space $\boldsymbol{H}_{\mathfrak{D}}$ is a Hilbert space with respect to pointwise addition and scalar multiplication and the inner product $\langle f, g\rangle=\sum_{n=0}^{\infty}(f(n), g(n))$. The unilateral shift U_+ is defined on $\boldsymbol{H}_{\mathfrak{D}}$ so that $(U_+f)(n)=\begin{cases} 0 & (n=0) \\ f(n-1) & (n>0)\end{cases}$, for f in $\boldsymbol{H}_{\mathfrak{D}}$. The operator U_+ is an isometry and its adjoint, the backward shift, satisfies $(U_+^*f)(n)=f(n+1)$ for f in $\boldsymbol{H}_{\mathfrak{D}}$. The sequence $\{U_+^{*n}\}$ converges strongly to 0. The minimal unitary extension U of U_+ is the bilateral shift defined on $\boldsymbol{L}_{\mathfrak{D}}$, where $\boldsymbol{L}_{\mathfrak{D}}$ is the space of functions f from the integers $\boldsymbol{Z}$ to $\mathfrak{D}$ so that $\sum_{n=-\infty}^{\infty}\|f(n)\|^2<\infty$ and U is defined $(Uf)(n)=f(n-1)$ for f in $\boldsymbol{L}_{\mathfrak{D}}$. It is easily verified that U is unitary and if we identify $\boldsymbol{H}_{\mathfrak{D}}$ as a subspace of $\boldsymbol{L}_{\mathfrak{D}}$ in the obvious way, then $U_+=U|\boldsymbol{H}_{\mathfrak{D}}$.

A result due to VON NEUMANN [6] states that every isometry V on $\mathfrak{H}$ is of the form $V=U_+\oplus V_0$ on $\mathfrak{H}=\boldsymbol{H}_{\mathfrak{D}}\oplus\mathfrak{H}_0$, where U_+ is the unilateral shift on $\boldsymbol{H}_{\mathfrak{D}}$ and V_0 is a unitary operator on $\mathfrak{H}_0$. Then $W=U\oplus V_0$ on $\mathfrak{K}=\boldsymbol{L}_{\mathfrak{D}}\oplus\mathfrak{H}_0$ is a unitary extension of V so that the smallest reducing subspace for W containing $\mathfrak{H}$ is $\mathfrak{K}$. This extension is unique to an isomorphism (cf. [3] or [9]). Let P denote the projection of $\mathfrak{K}$ onto $\mathfrak{H}$. As in the case of A_T we find it convenient to allow P to denote operators from $\mathfrak{H}$ to $\mathfrak{K}$ and $\mathfrak{K}$ to $\mathfrak{H}$ as well as from $\mathfrak{K}$ to $\mathfrak{K}$.

The following theorem reduces the solution of the equation $V_1^*XV_2=X$ for isometries V_1 and V_2 to the case of unitaries. In case $V_1=V_2$ a proof could be given based on a result of LEBOW [5, p. 68]. The following proof is based in part on a proof due to BROWN and HALMOS [1].

Theorem 2. *For $i=1, 2$, let V_i be an isometry on $\mathfrak{H}_i$ with minimal unitary extension W_i on $\mathfrak{K}_i$ and let P_i be the projection of $\mathfrak{K}_i$ onto $\mathfrak{H}_i$. Then $\mathfrak{S}(V_1, V_2)=$ $=P_1\mathfrak{S}(W_1, W_2)P_2$. Moreover, any X in $\mathfrak{S}(V_1, V_2)$ can be represented in the form $X=P_1YP_2$ with a Y in $\mathfrak{S}(W_1, W_2)$ such that $\|Y\|=\|X\|$.*

Proof. If B is an operator from $\mathfrak{K}_2$ to $\mathfrak{K}_1$ so that $W_1^* B W_2 = B$, then $V_1^* P_1 B P_2 V_2 = P_1 W_1^* P_1 B P_2 W_2 P_2 = P_1 W_1^* B W_2 P_2 = P_1 B P_2$ where the identities $P_1 W_1^* P_1 = P_1 W_1^*$ and $P_2 W_2 P_2 = W_2 P_2$ follow from the fact that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are invariant subspaces for W_1 and W_2, respectively. Thus $P_1 \mathfrak{S}(W_1, W_2) P_2$ is contained in $\mathfrak{S}(V_1, V_2)$. Note that $\|P_1 B P_2\| \leqq \|B\|$.

Conversely, suppose C is in $\mathfrak{S}(V_1, V_2)$; we want to define B from $\mathfrak{K}_2$ to $\mathfrak{K}_1$ so that $C = P_1 B P_2$ and $W_2 B W_1 = B$. The operator B will be obtained as the strong limit of the sequence $\{B_n\}$, where $B_n = W_1^{*n} P_1 C P_2 W_2^n$.

An elementary computation shows for $i = 1, 2$, that $P_{i,n} = W_i^{*n} P W_i^n$ is the projection of $\mathfrak{K}_i$ onto $W_i^{*n} \mathfrak{H}_i$ and that the sequence $\{P_{i,n}\}_n$ is monotonically increasing and converges strongly to the identity on $\mathfrak{K}_i$.

Observe now that since $\|B_n\| = \|W_1^{*n} P_1 C P_2 W_2^n\| \leqq \|C\|$, the sequence in uniformly bounded. Moreover, for $n \geqq m \geqq 0$ we have

$$P_{1,m} B_n P_{2,m} = W_1^{*m} P_1 W_1^m W_1^{*n} P_1 C P_2 W_2^n W_2^{*m} P_2 W_2^m =$$
$$= W_1^{*m} P_1 W^{*n-m} C W_2^{n-m} P_2 W_2^m = W_1^{*m} P_1 V_1^{*n-m} C V_2^{n-m} P_2 W_2^m =$$
$$= W_1^{*m} P_1 C P_2 W_2^m = B_m$$

so that $P_{1,m} B_n P_{2,m}$ is independent of n so long as n is greater than m. Thus for x in $P_{2,m}\mathfrak{K}_2$ and y in $P_{1,m}\mathfrak{K}_1$ we have

$$\lim_{n\to\infty} (B_n x, y) = \lim_{n\to\infty} (P_{1,m} B_n P_{2,m} x, y) = (B_m x, y).$$

Thus, $\lim\limits_{n\to\infty} (B_n x, y)$ exists for x in the dense subset $\bigcup\limits_m P_{2,m}\mathfrak{K}_2$ of $\mathfrak{K}_2$ and for y in the dense subset $\bigcup\limits_m P_{1,m}\mathfrak{K}_1$ of $\mathfrak{K}_1$. Since the sequence is uniformly bounded, we have that the sequence $\{B_n\}$ converges weakly to an operator B in $\mathfrak{L}(\mathfrak{K}_2, \mathfrak{K}_1)$. That $P_1 B P_2 = C$ and $W_1^* C W_2 = C$ are obvious. Thus we have completed the proof that $\mathfrak{S}(V_1, V_2)$ is equal to $P_1 \mathfrak{S}(W_1, W_2) P_2$.

In the preceding argument if we notice that we also have $B_n = P_{1,n} B P_{2,n}$, then using the fact that the sequences $\{P_{1,n}\}$ and $\{P_{2,n}\}$ converge strongly to the identity operators on $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively, we see that the sequence $\{B_n\}$ converges strongly to B, hence $\|B\| \leqq \|C\|$. From this it follows that any C in $\mathfrak{S}(V_1, V_2)$ can be represented in the form $C = P_1 B P_2$ with a B in $\mathfrak{S}(W_1, W_2)$ such that $\|B\| = \|C\|$.

4. We next study the space $\mathfrak{S}(W_1, W_2)$ for unitary operators W_1 and W_2 defined on the spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively. We begin with a lemma which is a mild generalization of a result due to PUTNAM [7]. We state the result for normal operators which necessitates the use of the Putnam—Fuglede Theorem. The same result for unitary operators has an elementary proof.

Lemma 4.1. *Let M and N be normal operators on the spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively, and let B an operator in $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ satisfying $BM = NB$. If $\mathfrak{M}$ denotes the orthogonal complement in $\mathfrak{H}$ of the kernel of B and $\mathfrak{N}$ denotes the closure in $\mathfrak{K}$ of the range of B, then $\mathfrak{M}$ reduces M, $\mathfrak{N}$ reduces N, and $M|\mathfrak{M}$ is unitarily equivalent to $N|\mathfrak{N}$.*

Proof. Let $B = PU$ be the polar decomposition for B with U a partial isometry in $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ and P a positive operator on $\mathfrak{K}$ so that the range of U is equal to $\mathfrak{N}$. Since $BM = NB$, the Putnam—Fuglede Theorem [8] implies $BM^* = N^*B$. These two equations imply that $\mathfrak{N}$ reduces N. Taking adjoints we have $M^*B^* = B^*N^*$ and $MB^* = B^*N$ which imply that $\mathfrak{M}$ reduces M.

Substituting we obtain $P^2N = BB^*N = BMB^* = NBB^* = NP^2$ so that P^2 commutes with N. Hence the positive square root of P^2 commutes with N so that $PUM = NPU = PNU$. This latter identity implies $UM = NU$ in view of the fact that the range of both UM and NU are contained in $\mathfrak{N}$ on which P is one-to-one. It now follows that $M|\mathfrak{M}$ and $N|\mathfrak{N}$ are unitarily equivalent with the isometry $U|\mathfrak{M}$ with range $\mathfrak{N}$ effecting this equivalence.

Returning to the situation of W_1 and W_2 unitary on $\mathfrak{K}_1$ and $\mathfrak{K}_2$ what we would like to do is to describe the space $\mathfrak{S}(W_1, W_2)$ more or less explicity. To do this, however, would take us too far afield. We content ourselves with determining when $\mathfrak{S}(W_1, W_2) \neq (0)$. Let $E(\delta)$ and $F(\delta)$ be the spectral measures for W_1 and W_2, respectively (cf. [3]). The unitary operators W_1 and W_2 are said to be *relatively singular* if the measures $\mu(\delta) = (E(\delta)x, x)$ and $\nu(\delta) = (F(\delta)y, y)$ are relatively singular for vectors x in $\mathfrak{K}_1$ and y in $\mathfrak{K}_2$.

Theorem 3. *If for $i = 1, 2$, W_i is a unitary operator on $\mathfrak{K}_i$, then $\mathfrak{S}(W_1, W_2) = (0)$ if and only if W_1 and W_2 are relatively singular.*

Proof. Suppose B is in $\mathfrak{S}(W_1, W_2)$ and $\mathfrak{M}$ and $\mathfrak{N}$ are defined as in the lemma. Then the operators $W_1|\mathfrak{M}$ and $W_2|\mathfrak{N}$ are unitarily equivalent. If U is an isometry from $\mathfrak{M}$ onto $\mathfrak{N}$ effecting this equivalence and x is any vector in $\mathfrak{M}$, then the measures $(E(\delta)x, x)$ and $(F(\delta)Ux, Ux)$ are identical. If $B \neq 0$, then $\mathfrak{M} \neq (0)$, so we can choose $x \neq 0$. Thus, $\mathfrak{S}(W_1, W_2) \neq (0)$ implies that W_1 and W_2 are not relatively singular.

If W_1 and W_2 are not relatively singular, then there exists vectors x in $\mathfrak{K}_1$ and y in $\mathfrak{K}_2$ so that the measures $\mu(\delta) = (E(\delta)x, x)$ and $\nu(\delta) = (F(\delta)y, y)$ are not relatively singular. Let $\mathfrak{M}_x$ and $\mathfrak{M}_y$ be the cyclic reducing subspaces generated by x and y foı the operators W_1 and W_2. It follows from the multiplicity theory for normal operators (cf. [3]) that there exist vectors x_0 in $\mathfrak{M}_x$ and y_0 in $\mathfrak{N}_y$ so that the measures $\mu_0(\delta) = (E(\delta)x_0, x_0)$ and $\nu_0(\delta) = (F(\delta)y_0, y_0)$ are mutually absolutely continuous. Thus the unitary operators $W_1|\mathfrak{M}_{x_0}$ and $W_2|\mathfrak{N}_{x_0}$ are unitarily equivalent. Let V be an isometry from $\mathfrak{M}_{x_0}$ onto $\mathfrak{N}_{y_0}$ so that $(W_1|\mathfrak{M}_{x_0}) = V^*(W_2|\mathfrak{N}_{y_0})V$. If we define the operator B in $\mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ so that $Bw = Vw$ for w in $\mathfrak{M}_{x_0}$ and $Bw = 0$ for w orthogonal to $\mathfrak{M}_{x_0}$, then B is in $\mathfrak{S}(W_1, W_2)$. Thus the proof is complete.

Implicit in the proofs of lemma 4.1 and the preceding theorem is a recipe for constructing the elements of $\mathfrak{S}(W_1, W_2)$. We will not elaborate on this.

5. We now consider the operator inequalities $T^*XT \leqq X$ and $T^*XT \geqq X$ for a given contraction T and unknown Hermitian operator X. We show first that we can restrict our attention to the first inequality and consider only positive solutions. Before stating this result we introduce the following terminology. A positive operator Q satisfying $T^*QT \leqq Q$ is said to be a *pure solution* if the sequence $\{T^{*n}QT^n\}$ converges strongly to 0 and we let $\mathfrak{Q}_T$ denote the set of pure positive solutions to $T^*QT \leqq Q$.

Theorem 4. *Let T be a contraction on $\mathfrak{H}$ and H (or K) be a Hermitian operator on $\mathfrak{H}$ so that $T^*HT \geqq H$ ($T^*KT \leqq K$). Then there exist Hermitian operators R and Q so that $H = R - Q$ ($K = R + Q$), $T^*RT = R$, $TQT^* \leqq Q$ and Q is pure. Moreover, this decomposition is unique.*

Proof. Since setting $H = -K$ reduces the second case to the first we consider only the case of a Hermitian H so that $T^*HT \geqq H$. Then the sequence $\{T^{*n}HT^n\}$ is a bounded monotonically increasing sequence of Hermitian operators. Thus it converges strongly to a Hermitian operator R. It is clear that $T^*RT = R$. Setting $Q = R - H$ we see that Q is positive and $T^*QT = T^*RT - T^*HT = R - T^*HT \leqq$ $\leqq R - H = Q$ or $T^*QT \leqq Q$. Moreover, since $T^{*n}QT^n = R - T^{*n}HT^n$ we see that Q is pure. Lastly, suppose $H = R_1 - Q_1$ with $T^*R_1T = R_1$, $T^*Q_1T \leqq Q_1$ and Q_1 is pure. Then $R_1 - R = T^{*n}(R_1 - R)T^n = T^{*n}(Q - Q_1)T^n$ and since the latter sequence converges strongly to 0 we have $R_1 = R$ and $Q_1 = Q$.

This result reduces the solution of the inequalities $T^*XT \geqq X$ and $T^*XT \leqq X$ to the study of the pure positive solutions to the latter inequality. This we shall do in two steps. Firstly, we characterize the pure positive solutions for $T^*QT \leqq Q$ using a construction due to Sz.-Nagy and Foiaş [9, p. 199] who used it for the case in which T is a co-isometry. This will reduce the study of this inequality to that of a commutation identity. Secondly, we make use of a recent result [10] of the same authors to study the obtained commutation identity.

Theorem 5. *Let T be a contraction on $\mathfrak{H}$. A positive operator Q on $\mathfrak{H}$ is a pure solution to $T^*QT \leqq Q$ if and only if there exists a unilateral shift U_+ on a space $\mathbf{H}_{\mathfrak{D}}$ and an operator C from $\mathfrak{H}$ to $\mathbf{H}_{\mathfrak{D}}$ so that $Q = C^*C$ and $CT = U_+^*C$.*

Proof. Suppose Q is a pure solution and set $R^2 = Q - T^*QT$. Then $Q = \sum_{n=0}^{\infty} T^{*n}(Q - T^*QT)T^n = \sum_{n=0}^{\infty} T^{*n}R^2T^n$, where the sum converges in the strong topology. We let $\mathfrak{D}$ be the closure of the range of R and consider U_+^* on $\boldsymbol{H}_{\mathfrak{D}}$. For x in $\mathfrak{H}$ the function f defined on $\boldsymbol{Z}^+$ so that $f(n)=RT^n x$ is in $\boldsymbol{H}_{\mathfrak{D}}$ since

$$\sum_{n=0}^{\infty}\|f(n)\|^2 = \sum_{n=0}^{\infty}(T^{*n}R^2T^n x, x) = \|Q^{1/2}x\|^2.$$

Moreover, the map from $\mathfrak{H}$ to $\boldsymbol{H}_{\mathfrak{D}}$ defined by $Cx=f$ is bounded, $(C^*Cx, x)= =\|Cx\|^2=\|f\|^2=\|Q^{1/2}x\|^2=(Qx, x)$ so that $Q=C^*C$ and $(CTx)(n)=RT^{n+1}x= =[U_*^+(Cx)](n)$ so that $CT=U_+^*C$. Thus the result is proved one way.

If U_*^+ is the backward shift on some $\boldsymbol{H}_{\mathfrak{D}}$ and C is an operator from $\mathfrak{H}$ to $\boldsymbol{H}_{\mathfrak{D}}$ so that $CT=U_+^*C$, then $T^*C^*CT=C^*U_+U_+^*C\leqq C^*C$ so that $Q=C^*C$ satisfies $T^*QT\leqq Q$. Further, $T^{*n}QT^n=C^*U_+^nU_+^{*n}C$ and the latter sequence converges strongly to 0 since $\{U_+^{*n}\}$ does.

We next state a recent result due to Sz.-Nagy and Foiaş [10] which determines the operators satisfying the conclusion of Theorem 5. Recall that for a contraction T on $\mathfrak{H}$, there exists a unique co-isometry V^* on a space $\mathfrak{K}_+$ containing $\mathfrak{H}$ so that $\mathfrak{H}$ is an invariant subspace for V^*, $T=V^*|\mathfrak{H}$, and the smallest reducing subspace for V^* containing $\mathfrak{H}$ is $\mathfrak{K}_+$; V is the minimal isometric dilation of T^* (cf. [9, p. 11]). Call V^* the canonical co-isometry of T. The minimal unitary extension W on $\mathfrak{K}$ of V is the minimal unitary dilation for T^*, that is, if P denotes the projection of $\mathfrak{K}$ onto $\mathfrak{H}$, then $T^{*n}=PW^n|\mathfrak{H}$ for all positive n and the smallest reducing subspace for W containing $\mathfrak{H}$ is $\mathfrak{K}$.

The theorem of Sz.-Nagy and Foiaş [10] can be stated (by taking adjoints) as follows.

Theorem 6. *For $i=1,2$, let T_i be a contraction on $\mathfrak{H}_i$ with canonical co-isometry V_i^* on $\mathfrak{K}_{i+}$. An operator C in $\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ satisfies $CT_1=T_2C$ if and only if there exists an operator D in $\mathfrak{L}(\mathfrak{K}_{1+}, \mathfrak{K}_{2+})$ so that $DV_1^*=V_2^*D$ and $C=D|\mathfrak{H}$. Moreover, D can be chosen such that $\|D\|=\|C\|$.*

Let us remark the following. Suppose T is an isometry in $\mathfrak{H}$ and let W denote its minimal unitary extension in $\mathfrak{K}$. Minimality means that

$$\mathfrak{K}=\bigvee_0^{\infty} W^{-n}\mathfrak{H}. \tag{1}$$

From $T^n=W^n|\mathfrak{H}$ we have $T^{*n}=P_{\mathfrak{H}}W^{*n}|\mathfrak{H}$ $(n\geqq 0)$; thus W^* is an isometric (in fact, unitary) dilation of T^*. Moreover, (1) shows that W^* is the *minimal* isometric dilation of T^*. Thus W is the canonical co-isometry of T, as asserted.

Applying Theorem 6 of Sz.-Nagy and Foiaş to the case of isometric T_1 and T_2 we get the following

Corollary 5. 1. *For $i=1,2$, let V_i be an isometry on $\mathfrak{H}_i$ with minimal unitary extension W_i on $\mathfrak{K}_i$. An operator C in $\mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ satisfies $V_1C=CV_2$ if and only* *if there exists B in $\mathfrak{L}(\mathfrak{K}_2, \mathfrak{K}_1)$ satisfying $W_1B = BW_2$ and $C = B|\mathfrak{H}_2$. Moreover B can be chosen such that $\|B\| = \|C\|$.*

This result also follows from Theorem 2.

As corollaries to Theorems 5 and 6 we obtain results analogous with those of §2 and 3.

Corollary 5.2. *Let T be a contraction on $\mathfrak{H}$ with canonical co-isometry V^* on $\mathfrak{K}_+$ and let $P_{\mathfrak{H}}^+$ be the projection of $\mathfrak{K}_+$ on $\mathfrak{H}$. Then $\mathfrak{Q}_T = P_{\mathfrak{H}}^+ \mathfrak{Q}_{V^*} P_{\mathfrak{H}}^+$. Moreover, every X in $\mathfrak{Q}_T$ can be represented in the form $P_{\mathfrak{H}}^+ Y P_{\mathfrak{H}}^+$ with Y in $\mathfrak{Q}_{V^*}$ such that $\|X\| = \|Y\|$.*

Proof. If Q is in $\mathfrak{Q}_{V^*}$, then

$$T^* P_{\mathfrak{H}}^+ Q P_{\mathfrak{H}}^+ T = P_{\mathfrak{H}}^+ V Q V^* P_{\mathfrak{H}}^+ \leqq P_{\mathfrak{H}}^+ Q P_{\mathfrak{H}}^+$$

so that $P_{\mathfrak{H}}^+ \mathfrak{Q}_{V^*} P_{\mathfrak{H}}^+$ is contained in $\mathfrak{Q}_T$.

If Q is in $\mathfrak{Q}_T$, then from Theorem 5 it follows that there exists a backward shift U_+^* on $H_{\mathfrak{D}}$ and an operator C in $\mathfrak{L}(\mathfrak{H}, H_{\mathfrak{D}})$ so that $CT = U_+^* C$ and $Q = C^*C$. From Theorem 6 of Sz.-Nagy and Foiaş (case $T_2^* = U_+^*$) we obtain an operator D in $\mathfrak{L}(\mathfrak{K}_+, H_{\mathfrak{D}})$ so that $DV^* = U_+^* D$ and $C = D|\mathfrak{H}$. Thus again using Theorem 5 we have D^*D is in $\mathfrak{Q}_{V^*}$. Moreover, $Q = C^*C = P_{\mathfrak{H}}^+ D^*D P_{\mathfrak{H}}^+$ and the proof is complete.

Corollary 5.3. *Let T be a contraction on $\mathfrak{H}$ with minimal unitary dilation W on $\mathfrak{K}$ and let $P_{\mathfrak{H}}$ denote the projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Then $\mathfrak{Q}_T = P_{\mathfrak{H}} \mathfrak{Q}_W P_{\mathfrak{H}}$. Moreover, every X in $\mathfrak{Q}_T$ can be represented in the form $P_{\mathfrak{H}} Z P_{\mathfrak{H}}$ with Z in $\mathfrak{Q}_W$ such that $\|X\| = \|Z\|$.*

Proof. If V^* is the canonical co-isometry on $\mathfrak{K}_+$ then by the preceding corollary $\mathfrak{Q}_T = P_{\mathfrak{H}}^+ \mathfrak{Q}_{V^*} P_{\mathfrak{H}}^+$. If D satisfyes $DV^* = U_+^* D$, then $VD^* = D^* U_+$ so that from Corollary 5.1 it follows that there exists E in $\mathfrak{L}(L_{\mathfrak{D}}, \mathfrak{K})$ so that $D^* = E|H_{\mathfrak{D}}$ and $EU = W^*E$. Thus we have $D^*D = P_{\mathfrak{D}_{\mathfrak{H}}} E^* P_{\mathfrak{D}_{\mathfrak{H}}}$ and $W^* E P_{\mathfrak{D}_{\mathfrak{H}}} E^* W = EUP_{\mathfrak{D}_{\mathfrak{H}}} U^* E^* \leqq$ $\leqq EP_{\mathfrak{D}_{\mathfrak{H}}} E^*$ so that $EP_{\mathfrak{D}_{\mathfrak{H}}} E^*$ is in $\mathfrak{Q}_W$ and $\mathfrak{Q}_T$ is seen to be contained in $P_{\mathfrak{H}} \mathfrak{K}_W P_{\mathfrak{H}}$.

Conversely, if Q is in $\mathfrak{Q}_W$ and if R denotes the projection of $\mathfrak{K}$ onto $\mathfrak{K}_+$, then $VRQRV^* = RW^*RQRWR = RW^*QWR \leqq RQR$ so that RQR is in $\mathfrak{Q}_{V^*}$. Using the preceding corollary we have $P_{\mathfrak{H}} Q P_{\mathfrak{H}} = P_{\mathfrak{H}} RQR P_{\mathfrak{H}}$ is in $\mathfrak{Q}_T$ and the proof is complete.

Implicit in the preceding proof is a characterization of the operators in $\mathfrak{Q}_W$ for a unitary operator W. We state it without further proof.

Corollary 5.4. *Let W be a unitary operator on $\mathfrak{K}$. Then Q is in $\mathfrak{Q}_W$ if and only if there exists a Hilbert space $\mathfrak{D}$ and an operator E in $\mathfrak{L}(\mathfrak{K}, L_{\mathfrak{D}})$ so that $EW = UE$ and $Q = E^* P_{H_{\mathfrak{D}}} E$.*

We illustrate how the preceding results can be applied to obtain a result due to PUTNAM [8, Theorem 2. 3. 2]. Before stating it we need to recall the following. If W is a unitary operator on $\mathfrak{K}$ with spectral measure $E(\delta)$, then W is said to be *absolutely continuous* [*singular*] if the measure $\mu(\delta)=(E(\delta)x, x)$ is absolutely continuous [singular] for each vector x in $\mathfrak{K}$. If W is a unitary operator on $\mathfrak{K}_1$ then $\mathfrak{K} = \mathfrak{K}_a \oplus \mathfrak{K}_s$, where $\mathfrak{K}_a$ and $\mathfrak{K}_s$ are reducing subspaces for W so that $W|\mathfrak{K}_a$ is absolutely continuous while $W|\mathfrak{K}_s$ is singular. The operator $W|\mathfrak{K}_a$ is said to be the absolute continuous part of W. (See [3] for details and proofs.)

Corollary 5. 5. *Let W be a unitary operator on $\mathfrak{H}$ and Q be a pure positive solution to $W^*QW \leqq Q$. Then the range of Q is contained in the absolutely continuous part of W.*

Proof. From the preceding theorem it follows that there exists a backward shift U_+ on some $H_\mathfrak{D}$ and an operator C from $\mathfrak{H}$ to $H_\mathfrak{D}$ so that $Q=C^*C$ and $CW= =U_+^*C$. Thus there exists by Corollary 5. 1 an operator D from $\mathfrak{H}$ to $L_\mathfrak{D}$ so that $D=C^*|H_\mathfrak{D}$ and $W^*D=DU$. Moreover, since $Q=C^*C$, the closure of the range of Q is equal to the closure of the range of C^* which in turn is equal to the closure of $DH_\mathfrak{D}$. Thus our problem is reduced to showing that $DH_\mathfrak{D}$ is contained in the absolutely continuous part of W.

Using lemma 4. 1 we have that W restricted to the closure of the range of D is unitarily equivalent to U restricted to the orthogonal complement of the kernel of D. The latter unitary operator is a part of the bilateral shift and so must be absolutely continuous. (We can compute the spectral measure in this case.) Thus $DH_\mathfrak{D}$ is contained in the absolutely continuous part of W and the proof is complete.

Corollary 5. 6. *Let W be a singular unitary operator on $\mathfrak{H}$ and H be a Hermitian operator on $\mathfrak{H}$ so that $W^*HW \geqq H$. Then W commutes with H.*

Proof. From Theorem 4 we have that $H = R-Q$ where R commutes with H and Q is a pure positive solution to $W^*QW \leqq Q$. From the preceding corollary we have the range of Q is contained in the absolutely continuous part of W which in this case has been assumed to be (0). Thus $Q=0$ and the proof is complete.

Recall that an operator T on $\mathfrak{H}$ is said to be hyponormal if $T^*T \geqq TT^*$ and completely non normal if for no subspace $\mathfrak{M}$ reducing T is $T|\mathfrak{M}$ normal.

Corollary 5. 7. *If T is an invertible completely non normal hyponormal operator on $\mathfrak{H}$ with polar decomposition $T=PU$, then U is absolutely continuous.*

Proof. Since T is invertible, the operator U is unitary and $U^*P^2U = =T^*T \geqq TT^* = P^2$. Thus from Theorem 4 it follows that $P^2=R-Q$, where R and Q are positive, U commutes with R and Q is a pure solution to $U^*QU \leqq Q$. Thus from the preceding corollary it follows that the range of Q is contained in the absolutely continuous part of U. If E is the spectral projection for U onto the singular part of U, then $EP^2 = ER + EQ = ER = RE = P^2E$. Thus $T|E\mathfrak{H} = (EPE)(EUE)|E\mathfrak{H}$, where EPE is positive, EUE is unitary and EPE commutes with EUE. Thus $T|E\mathfrak{H}$ is normal implying by hypothesis that $E=0$ and the proof is complete.

This is related to the result that every compact hyponormal operator is normal (cf. [4]). We offer a proof of this result in § 6.

We conclude this section with a further remark concerning the inequality $T^*XT \geqq X$ for positive operators X. In Theorem 1 we showed that solutions for the equation $T^*XT = X$ could be obtained from solutions to $V_T^*XV_T = X$ where V_T is the isometry associated with the contraction T. This isometry is only part of the minimal unitary dilation for T to which we reduced the study of $T^*XT \leqq X$. It is therefore of interest that the study of $T^*XT \geqq X$ can be reduced to that of $V_T^*XV_T \geqq X$.

For T a contraction let $\mathfrak{P}_T$ denote the class of positive operators P so that $T^*PT \geqq P$.

Theorem 7. *Let T be a contraction on $\mathfrak{H}$ and A_T and V_T as in Theorem 1. Then $\mathfrak{P}_T = (0)$ if and only if $A_T = 0$ and $\mathfrak{P}_T = A_T \mathfrak{P}_{V_T} A_T$. Moreover, every X in $\mathfrak{P}_T$ can be represented in the form $A_T Y A_T$ with Y in $\mathfrak{P}_{V_T}$ such that $\|X\| = \|Y\|$.*

The proof is the same as that of Theorem 1.

6. We now obtain some special results in the presence of a compactness hypothesis. Before we can state our result we need a lemma concerning the subspace $\mathfrak{U}_T$ spanned by the eigenvectors of a contraction which belong to an eigenvalue of modulus one. See [9, pp. 8—9] for the proof.

Lemma 6. 1. *If T is a contraction on $\mathfrak{H}$, then $\mathfrak{U}_T$ reduces T and $T|\mathfrak{U}_T$ is a unitary operator with pure point spectrum.*

Our main result in this section is the following.

Theorem 8. *Let T be a contraction on $\mathfrak{H}$. If Q is a compact positive operator in $\mathfrak{P}_T$, then Q is in $\mathfrak{S}_T$. Further, if A is a compact operator in $\mathfrak{S}_T$, then A and A^* commute with T, $\mathfrak{U}_T$ reduces A, and $A|\mathfrak{U}_T^{\perp} = 0$.*

Proof. Suppose Q is positive, compact, and so that $T^*QT \geqq Q$. Let $\lambda_1 > \lambda_2 > \cdots$ be the non zero eigenvalues of Q, and let $\mathfrak{I}_1, \mathfrak{I}_2, \cdots$ be the corresponding eigenspaces. Each of these eigenspaces is finite dimensional and, denoting by P_n the (orthogonal) projection of $\mathfrak{H}$ onto $\mathfrak{I}_n$, we have

$$Qx = \sum_n \lambda_n P_n x \quad \text{for all} \quad x \in \mathfrak{H}.$$

We shall prove that each $\mathfrak{I}_n$ reduces T, and that $T|\mathfrak{I}_n$ is unitary. We do this by induction on n. Suppose this is true for all n less than some $m (\geqq 1)$ (for $m = 1$

this hypothesis being void). For x in $\mathfrak{J}_m$, Tx satisfies then the condition of being orthogonal to each $\mathfrak{J}_n$ with $n<m$ (this condition being void if $m=1$); so we have

$$\lambda_m \|Tx\|^2 \geqq (QTx, Tx) = (T^*QTx, x) \geqq (Qx, x) = \lambda_m \|x\|^2.$$

Since T is a contraction, this implies $\|Tx\|=\|x\|$ and $(QTx, Tx)=\lambda_m\|Tx\|^2$. Thus $T\mathfrak{J}_m \subset \mathfrak{J}_m$, and $T_m = T|\mathfrak{J}_m$ is an isometry. However, since $\mathfrak{J}_m$ is finite dimensional it follows that T_m is unitary. Then so is T_m^*, which is equal to $P_m T^*|\mathfrak{J}_m$. So we have for x in $\mathfrak{J}_m$

$$\|x\| = \|T_m^* x\| = \|P_m T^* x\| \leqq \|T^* x\| \leqq \|x\|,$$

and this implies $P_m T^* x = T^* x$. Hence $T^*\mathfrak{J}_m \subset \mathfrak{J}_m$ so that $\mathfrak{J}_m$ reduces T.

So we have shown that each $\mathfrak{J}_n$ $(n=1, 2, \cdots)$ reduces T to a unitary operator. It follows for an arbitrary x in $\mathfrak{H}$

$$T^* P_n T x = T^* T P_n x = P_n x \qquad (n=1, 2, \cdots),$$

and hence

$$T^* Q T x = \sum_n \lambda_n T^* P_n T x = \sum_n \lambda_n P_n x = Qx.$$

Thus Q is in $\mathfrak{S}_T$.

Consider now a compact operator A in $\mathfrak{S}_T$.

If $T^*AT=A$, taking adjoints we obtain $T^*A^*T=A^*$ so that if $A = H+iK$ are the real and imaginary parts of A, then $T^*HT=H$ and $T^*KT=K$. Thus the proof can be reduced to the case of a Hermitian operator.

If H is Hermitian, then there exists a reducing subspace $\mathfrak{N}$ for H so that $H_1 = H|\mathfrak{N}$ and $H_2 = -H|\mathfrak{N}^\perp$ are positive operators. Substituting we obtain the equation $T^*H_1T - T^*H_2T = H_1 - H_2$, where $T^*H_1T \geqq 0$ and $T^*H_2T \geqq 0$.

If R denotes the projection of $\mathfrak{H}$ onto $\mathfrak{N}$, then

$$(TR)^* H_1 (TR) \geqq (TR)^* H_1 (TR) - (TR)^* H_2 (TR) = RH_1R - RH_2R = H_1$$

so that $(TR)^*H_1(TR) \geqq H_1$. Since H_1 is positive and compact it follows from the above that $\mathfrak{U}_{TR}$ reduces H_1 and $H_1|\mathfrak{U}_{TR}^\perp = 0$. If x is in $\mathfrak{U}_{TR}$, then for some $e^{i\theta}$ we have $TRx = e^{i\theta}x$ so that $\|x\| = \|TRx\| \leqq \|Rx\| \leqq \|x\|$. Thus $Rx = x$ which implies $Tx = e^{i\theta}x$ and x is in $\mathfrak{U}_T$. Hence $\mathfrak{U}_{TR} \subset \mathfrak{U}_T$ so that $\mathfrak{U}_T$ reduces H_1 and $H_1|\mathfrak{U}_T^\perp = 0$.

Consideration of the identity $T^*(-H)T = (-H)$ yields the corresponding results for H_2. Thus $\mathfrak{U}_T$ reduces H and $H|\mathfrak{U}_T^\perp = 0$. Moreover, since $TT^*|\mathfrak{U}_T$ is the identity on $\mathfrak{U}_T$, we obtain $HT = TT^*HT = TH$ and $T^*H = T^*HTT^* = HT^*$. This completes the proof.

A lemma of Dye [2, lemma 3. 1] is an immediate corollary to Theorem 8.

The result of Brown and Halmos concerning compact Toeplitz operators [1] admits the following generalization.

Corollary 6. 1. *If T is a contraction on $\mathfrak{H}$ with no eigenvalues of modulus one and A is a compact operator in $\mathfrak{S}_T$, then $A=0$.*

The following corollary is well known (cf. [4]).

Corollary 6. 2. *If T is a compact hyponormal operator, then T is normal.*

Proof. If $T=QV$ is the polar decomposition for T, then Q is positive and compact. Further, $V^*Q^2V \geqq Q^2$. Theorem 8 now applies to conclude $V^*Q^2V=Q^2$ so that $T^*T=TT^*$ and T is normal.

Corollary 6. 3. *If T is a contraction on $\mathfrak{H}$ so that A_T is compact, then A_T is a finite dimensional projection and $T|A_T\mathfrak{H}$ is unitary.*

Proof. From the definition of A_T it follows that $T^*A_T^2T=A_T^2$. Thus by Theorem 8 we see that $T|A_T^2\mathfrak{H}$ is unitary so that for x in $\mathfrak{H}$ we obtain $\|A_T^2x\| = \lim_{n\to 0}\|T^nA_Tx\|^2=\|A_Tx\|$. Since A_T is a positive contraction we obtain $A_T^2=A_T$. Therefore A_T is a compact projection which implies it is finite dimensional.

We next state a couple of miscellaneous corollaries. Recall that for operators V and W defined on $\mathfrak{H}$ and $\mathfrak{K}$, W is said to be a quasi-affine transform of V is there exists a quasi-affinity S in $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, that is, an S with dense range and no null space, so that $VS=SW$ (cf. [9]).

Corollary 6. 4. *If the contraction K on $\mathfrak{H}$ is the quasi-affine transform of the isometry V on $\mathfrak{H}$, where $SK=VS$, and SK is compact, then K and V are unitary and unitarily equivalent.*

Proof. Since SKK^*S^* is positive and compact and $V^*SKK^*S^*V = V^*VSS^*V^*V=SS^* \geqq SKK^*S^*$, we can apply Theorem 8 to conclude that $S(I-KK^*)S^*=0$. Since S and S^* have no null space we conclude that K^* is an isometry. Lastly, since $VS=SK$ has no null space, neither can K which implies K is unitary. Thus SK has dense range which implies V is unitary. An application of Lemma 4. 1 completes the proof.

We now remark that the preceding corollary contains two different results. Firstly, in order for a compact contraction to be the quasi-affine transform of an isometry, the underlying space must be finite dimensional. Secondly, in order for a contraction to be the quasi-affine transform of an isometry with a compact operator implementing this equivalence, both the contraction and the isometry must be unitary.

Corollary 6. 5. *Let S and T be contractions on $\mathfrak{H}$ and A be a Hermitian compact operator on $\mathfrak{H}$ so that $S^*AT=A$. If $\mathfrak{M}=\overline{A\mathfrak{H}}$, then $S|\mathfrak{M}=V_1$ and $T|\mathfrak{M}=V_2$ are unitary, $V_1=V_2$, and V_1 and V_2 commute with $A|\mathfrak{M}$.*

Proof. From $S^*AT=A$ it follows that $S^*A^2S \geqq S^*ATT^*AS=A^2$ so that it follows from Theorem 8 that S commutes with A^2 and $\mathfrak{M}\subset\mathfrak{U}_S$. Similarly, consideration of the identity $T^*AS=A$ leads to the fact that T commutes with A^2 and $\mathfrak{M}\subset\mathfrak{U}_{T^*}=\mathfrak{U}_T$. The result now follows.

7. We conclude with a few remarks. In this paper we have been considering the equation $S^*XT=X$ and the inequalities $T^*XT\geqq X$ and $T^*XT\leqq X$ for contractions S and T. The assumption that S and T be contractions is crucial for results of this nature to hold.

If T is a contraction on $\mathfrak{H}$ not having 1 as an eigenvalue, then the Cayley transform $A=(I+T)(I-T)^{-1}$ of T can be defined. The set of operators obtained in this manner is the class of maximal accretive operators (cf. [9]). Recall that a densely defined operator A on $\mathfrak{H}$ is said to be *accretive* if $\operatorname{Re}(Ax, x)\geqq 0$ for x in the domain of A, and *maximal accretive* if no proper extension of A is accretive.

If S and T are contractions on $\mathfrak{H}$ and $\mathfrak{K}$ with Cayley transforms A and B, then for X in $\mathfrak{L}(\mathfrak{K}, \mathfrak{H})$ the equation $S^*XT=X$ holds if and only if $B^*X=-XA$. Thus this equation is amenable to the technique of §§ 2 and 3 for accretive operators A and B. The inequalities $T^*XT\geqq X$ and $T^*XT\leqq X$ for X Hermitian become $A^*X+XA\geqq 0$ and $A^*X+XA\leqq 0$ and can be solved with the results of § 5. The results of the rest of the paper have similar interpretations in terms of accretive operators.

Further, these results have extensions to one parameter semi-groups of contractions and indeed to other commutative semi-groups of contractions, but we will not pursue them.

Lastly, we conclude with an example. Recall that if N is a normal operator on $\mathfrak{K}$ and $\mathfrak{H}$ is an invariant subspace for N, then the operator $T=N|\mathfrak{H}$ is said to be *subnormal*. If the smallest reducing subspace for N containing $\mathfrak{H}$ is $\mathfrak{K}$, then N is said to be the *minimal normal extension of* T. This is unique to an isomorphism (cf. [4]). In case N is unitary, then T is an isometry and an isometry is subnormal by our previous remarks.

Corollary 5. 1 can be interpreted as stating that all "commuting maps" between isometries "lift" to their minimal normal extensions. We want to show that this is not true for subnormal operators in general. We first prove the following lemma.

Lemma 7. 1. *For $i=1,2$, let T_i be a subnormal operator on $\mathfrak{H}_i$ having minimal normal extension N_i on $\mathfrak{K}_i$. Let A be a quasi-affinity in $\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ so that $T_2A=AT_1$. Then a necessary condition that there exist B in $\mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ so that $N_2B=BN_1$ and $A=B|\mathfrak{H}_1$ is for N_1 and N_2 to be unitarily equivalent.*

Proof. Suppose such an operator B exists. Then as in the proof of lemma 4. 1, the closure $\mathfrak{N}$ of the range of B reduces N_2. If P denotes the projection of $\mathfrak{K}_2$ onto $\mathfrak{K}_2\ominus\mathfrak{N}$, then for a dense set of x in $\mathfrak{H}_2$ there exists y in $\mathfrak{H}_1$ so that $Ay=x$ and we have $Px=PAy=PBy=0$. Thus $\mathfrak{H}_2$ is contained in $\mathfrak{N}$ which contradicts the minimality of N_2 unless $\mathfrak{N}=\mathfrak{K}_2$. Similarly, the closure of the range of B^* must be $\mathfrak{K}_1$. From lemma 4. 1 it follows that N_1 and N_2 are unitarily equivalent.

Corollary 7. 2. *There exist subnormal operators T_1 on $\mathfrak{H}_1$ and T_2 on $\mathfrak{H}_2$ with minimal normal extensions N_1 on $\mathfrak{K}_1$ and N_2 on $\mathfrak{K}_2$, respectively, and an operator A* in $\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ *satisfying $AT_1=T_2A$ for which there is no B in* $\mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ *satisfying $BN_1=N_2B$ and $A=B|\mathfrak{K}_1$.*

Proof. There is an example in [4] due to SARASON of similar subnormal operators T_1 on $\mathfrak{H}_1$ and T_2 on $\mathfrak{H}_2$ so that their minimal normal extensions N_1 on $\mathfrak{K}_1$ and N_2 on $\mathfrak{K}_2$ are not unitarily equivalent. If A is the invertible operator so that $T_2A=AT_1$, then if follows from the preceding lemma that there exists no B in $\mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ satisfying $N_2B=BN_1$ and $A=B|\mathfrak{H}_1$.

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On the cosine of unbounded operators

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The concept of the cosine $\cos_R A$ of an accretive linear operator A is a useful parameter in determining when the product of two accretive operators is itself accretive (see [2]). To be (nontrivially) applicable in that context, it is necessary that the cosine be strictly positive, which is always the case for strongly accretive bounded operators. The object of the present note is to show that $\cos_R A \equiv 0$ for all unbounded accretive operators A. This fact seems interesting since it serves to distinguish geometrically the topological notions of boundedness and unboundedness for strongly accretive operators.

We restrict ourselves to (real or complex, separable or non-separable) pre-Hilbert spaces H and to unbounded accretive operators A. It is not necessary that A be closed or densely defined, and no completeness properties for H are needed; however, it should be noted that when H is a Hilbert space and $D(A)$, the domain of A, is dense, then the accretiveness of A implies that A is closeable (see [3, p. 268]). Our demonstration does not immediately extend to Banach spaces because we make use of both bilinearity of the inner product (not generally present for the semi-inner product, see [1, 4]) and orthogonality in H.

We recall that an operator A is accretive if $\operatorname{Re}(Ax, x) \geqq 0$ for all $x \in D(A)$.

Definition.

$$\cos_R A = \inf_x \frac{\operatorname{Re}(Ax, x)}{\|Ax\| \cdot \|x\|}, \quad |\cos| A = \inf_x \frac{|(Ax, x)|}{\|Ax\| \cdot \|x\|} \qquad (x \in D(A),\ Ax \neq 0).$$

Theorem. $\cos_R A = 0$ *for all accretive unbounded operators* A.

Proof. If $\operatorname{Re}(Ax, x)$ is bounded above uniformly, $\cos A = 0$ immediately by the unboundedness of A; therefore we may assume that there exists a sequence $\{u_n\}$, $\|u_n\| = 1$, $\operatorname{Re}(Au_n, u_n) \to \infty$. Let $w_n = \eta_n u_n + \xi_n v_n$ where $\eta_n = [\operatorname{Re}(Au_n, u_n)]^{-\alpha}$, $\xi_n = (1 - \eta_n^2)^{\frac{1}{2}}$, $\frac{1}{2} \leqq \alpha < 1$, and $v_n \in D(A)$, $\|v_n\| = 1$; v_n will be specifically chosen

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later. Then for all sufficiently large n, if $\|Av_n\|$ is uniformly bounded, one has by the inverse triangle inequality:

$$R(w_n) = \frac{\mathrm{Re}\,(Aw_n, w_n)}{\|Aw_n\|\cdot\|w_n\|} \leqq \tag{1}$$

$$\leqq \frac{\xi_n^2\,\mathrm{Re}\,(Av_n, v_n) + \xi_n\eta_n\,\mathrm{Re}\,(Av_n, u_n) + \eta_n\xi_n\,\mathrm{Re}\,(Au_n, v_n) + \eta_n^2\,\mathrm{Re}\,(Au_n, u_n)}{[\eta_n\|Au_n\| - \xi_n\|Av_n\|]\cdot|\xi_n - \eta_n|} =$$

$$= (N_1 + N_2 + N_3 + N_4)/D,$$

with the denominator $D \to \infty$, since $|\xi_n - \eta_n| \to 1$, and $\|Au_n\|\cdot[\mathrm{Re}\,(Au_n, u_n)]^{-\alpha} \to \infty$ for $0 < \alpha < 1$; the latter may be seen as follows. Let $\|u_n\| = 1, \alpha < 1$, and $\mathrm{Re}\,(Au_n, u_n) \to \infty$. Then by SCHWARZ's inequality one has $\|Au_n\|^2\cdot[\mathrm{Re}\,(Au_n, u_n)]^{-2\alpha} \geqq [\mathrm{Re}\,(Au_n, u_n)]^{2-2\alpha} \to \infty$.

Let us now consider the four terms N_i/D separately. If $\|Av_n\|$ is uniformly bounded, clearly (by SCHWARZ's inequality) $N_1/D \to 0$ and $N_2 \to 0$. Also, $N_4 = 1$ if $\alpha = 1/2$, $N_4 \to 0$ if $\alpha > 1/2$; thus $(N_1 + N_2 + N_4)\cdot D^{-1} \to 0$ for $1/2 \leqq \alpha < 1$. Therefore if $|N_3|$ is uniformly bounded, $R(w_n) \to 0$ in (1). Now, if there exists at least one nontrivial vector $v \in D(A) \cap D(A^*)$ (let it have norm $= 1$), then taking $v_n \equiv v$, $\|Av_n\|$ is obviously uniformly bounded, and $|N_3| = \eta_n\xi_n\,|\mathrm{Re}(u_n, A^*v_n)| \leqq \|A^*v\|$. If $D(A) \cap D(A^*) = \{0\}$, we may proceed as follows. Select $x, y \in D(A)$, $\|x\| = \|y\| = 1$, $(x, y) = 0$, and let $v_n = \alpha_n x + \beta_n y$, $|\alpha_n|^2 + |\beta_n|^2 = 1$. Now choose α_n, β_n so that $(v_n, Au_n) = 0$; that this can always be done is assured by taking α_n and β_n from the solutions of the equation $\alpha_n(x, Au_n) + \beta_n(y, Au_n) = 0$. Then $\|v_n\| = 1$, $\|Av_n\| \leqq \|Ax\| + \|Ay\|$, $N_3 = 0$, and $R(w_n) \to 0$.

One may obtain the following stronger result.

Corollary. $|\cos| A = 0$ *for all unbounded operators.*

Proof. Replace $\mathrm{Re}\,(Ax_1, x_2)$ by $|(Ax_1, x_2)|$ everywhere in the above.

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A note on invariant linear forms on von Neumann algebras

By I. KOVÁCS and J. SZŰCS in Szeged

1. Let $\mathbf{A}$ be a von Neumann algebra [1]) in a complex Hilbert space $\mathfrak{H}$. We shall denote by $\mathbf{A}^+$ (resp. $\mathbf{A}_1$) the positive portion (resp. the unit ball) of $\mathbf{A}$: $\mathbf{A}^+ = \{T \in \mathbf{A}: T \geqq O\}$ (resp. $\mathbf{A}_1 = \{T \in \mathbf{A}: \|T\| \leqq 1\}$. If $\mathscr{G}$ is a group of $*$-automorphisms of $\mathbf{A}$, set $\mathbf{A}^{\mathscr{G}} = \{T \in \mathbf{A}: \theta(T) = T$ for each $\theta \in \mathscr{G}\}$. It is not hard to see that $\mathbf{A}^{\mathscr{G}}$ is a von Neumann subalgebra of $\mathbf{A}$. For a given pair $\mathbf{A}, \mathscr{G}$, we shall denote by $\mathscr{R}(\mathbf{A}, \mathscr{G})$ the space of all ultra-weakly continuous linear forms σ on $\mathbf{A}$ which are invariant with respect to $\mathscr{G}$. [2]) $\mathscr{R}^r(\mathbf{A}, \mathscr{G})$ (resp. $\mathscr{R}^+(\mathbf{A}, \mathscr{G})$) will denote the set of all real (resp. positive) [3]) elements of $\mathscr{R}(\mathbf{A}, \mathscr{G})$.

Consider an arbitrary element σ of $\mathscr{R}^r(\mathbf{A}, \mathscr{G})$ and set $M = \underset{T \in \mathbf{A}_1^+}{\mathrm{l.u.b.}}\, \sigma(T)$, where $A_1^+ = \mathbf{A}_1 \cap \mathbf{A}^+$. According to [4], there exists a projection E in $\mathbf{A}_1^+$ such that $\sigma(E) = M$. In this note we intend first to prove that E can be chosen to be an element of $\mathbf{A}^{\mathscr{G}}$. Then we shall study some consequences of this fact. Let us formulate it in the form of a

Theorem. *Let σ be an arbitrary element of $\mathscr{R}^r(\mathbf{A}, \mathscr{G})$ and set $M = \underset{T \in \mathbf{A}_1^+}{\mathrm{l.u.b.}}\, \sigma(T)$. Then there exists a projection F of $\mathbf{A}^{\mathscr{G}}$ such that $\sigma(F) = M$.*

Remark. If $\mathscr{G}$ is abelian, the theorem is a consequence of the Kakutani—Markov fixed point theorem (cf. [2, V. 10. 6]). In case A is G-finite ([3]), the support [4]) of $E^{\mathscr{G}}$ has the property required (cf. [4]).

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[1]) For the theory of von Neumann algebras we refer the reader to [1].

[2]) I. e. $\sigma(\theta(T)) = \sigma(T)$ for every $T \in \mathbf{A}$ and $\theta \in \mathscr{G}$.

[3]) For any $\sigma \in \mathscr{R}(\mathbf{A}, \mathscr{G})$, put $\sigma^*(T) = \overline{\sigma(T^*)}$. It is evident that $\sigma^* \in \mathscr{R}(\mathbf{A}, \mathscr{G})$. σ is said to be *real* (resp. *positive*) if $\sigma = \sigma^*$ (resp. $\sigma(T) \geqq 0$ for $T \in \mathbf{A}^+$). Any $\sigma \in \mathscr{R}(\mathbf{A}, \mathscr{G})$ can be uniquely written in the form $\sigma = \sigma_1 + i\sigma_2$ with $\sigma_1, \sigma_2 \in \mathscr{R}^r(\mathbf{A}, \mathscr{G})$. In fact, $\sigma_1 = \frac{1}{2}(\sigma + \sigma^*)$ and $\sigma_2 = \frac{1}{2i}(\sigma - \sigma^*)$. In the following we prefer to use the notations: $\sigma_1 = \mathrm{Re}\,\sigma$, $\sigma_2 = \mathrm{Im}\,\sigma$.

[4]) Cf. [1, App. III].

Proof of the theorem. Actually, the proof is based upon the following simple fact: Let $T_1, \cdots, T_n$ $(n=1, 2, \cdots)$ be elements in $\mathbf{A}^+$ with supports $E_1, \cdots, E_n$, respectively. Then the support E of $T=T_1+\cdots+T_n$ is equal to $\text{l.u.b.}_{1\leqq i\leqq n} E_i$. It is enough to show this statement for $n=2$, since then an induction argument will take care of the general case. For $n=2$ we prove that $I-E=(I-E_1)\cap(I-E_2)$ [5]) which evidently implies the assertion. Now the inequality $(I-E_1)\cap(I-E_2)\leqq I-E$ is evident. To prove the converse of it, consider an arbitrary element x of $(I-E)\mathfrak{H}$. Then $Tx=(T_1+T_2)x=0$. Thus, $(T_1+T_2x|x)=0$ which gives that $(T_1x|x)=$ $=-(T_2x|x)$. As $T_1, T_2\in\mathbf{A}^+$, this is possible only if $(T_1x|x)=(T_2x|x)=0$. But this implies that $T_1^{\frac{1}{2}}x=T_2^{\frac{1}{2}}x=0$, i.e. $T_1x=T_2x=0$. Hence $I-E\leqq(I-E_1)\cap$ $\cap(I-E_2)$, which was to be proved. Now let $E\in\mathbf{A}_1^+$ be a projection such that $\sigma(E)=M$ (cf. [4]), and consider an arbitrary finite system $\theta_1, \cdots, \theta_n$ of elements of $\mathcal{G}$. Set $S_n=\frac{1}{n}\sum_{i=1}^{n}\theta_i(E)$. It is evident that $S_n\in\mathbf{A}_1^+$ and $\sigma(S_n)=$ $=\frac{1}{n}\sum_{i=1}^{n}\sigma(\theta_i(E))=\sigma(E)=M$. Since, for every i $(1\leqq i\leqq n)$, $\theta_i(E)$ is a projection, the support F_n of S_n is equal to $\text{l.u.b.}_{1\leqq i\leqq n}\theta_i(E)$. Since $\sigma(S_n)=M$, we obtain that $\sigma(F_n)=M$, too (cf. [4]). Now if for each possible finite system J of elements of $\mathcal{G}$ we take the corresponding projection F_J, we obtain an upward directed family $\{F_J\}_{J\subset\mathcal{G}}$ of projections. Put $F=\text{l.u.b.}_{J\subset\mathcal{G}} F_J$. F is a cluster point of the projections F_J in the strong operator topology (cf. [1, App. III]). Now taking into account the topological properties of the elements of $\mathcal{G}$ and σ (cf. [1, chap. I. §§ 3—4]), for every $\theta_0\in\mathcal{G}$ we have $\theta_0(F)=\theta_0(\text{l.u.b.}_{J\subset\mathcal{G}} F_J)=\text{l.u.b.}_{J\subset\mathcal{G}}\theta_0(F_J)=\text{l.u.b.}_{J'\subset\mathcal{G}} F_{J'}$ [6]) $=F$, i.e. $F\in\mathbf{A}^{\mathcal{G}}$, and $\sigma(F)=\text{l.u.b.}_{J\subset\mathcal{G}}\sigma(F_J)$. But, by construction, $\sigma(F_J)=M$ for each $J(\subset\mathcal{G})$, thus $\sigma(F)=M$, qu.e.d.

2. Now let us present some consequences of this theorem.

Let σ_1 and σ_2 be two arbitrary elements of $\mathcal{R}(\mathbf{A}, \mathcal{G})$ such that for every $T\in\mathbf{A}^{\mathcal{G}}$ we have $\sigma_1(T)=\sigma_2(T)$. Put $\sigma=\sigma_1-\sigma_2$, $\sigma'=\text{Re}\,\sigma$ and $\sigma''=\text{Im}\,\sigma$. Since $\sigma'(T)=$ $=\sigma''(T)=0$ for every $T\in\mathbf{A}^{\mathcal{G}}$, by the preceding theorem we obtain that $\text{l.u.b.}_{S\in\mathbf{A}_1^+}\sigma'(S)=$ $=\text{l.u.b.}_{S\in\mathbf{A}_1^+}\sigma''(S)=0$. Arguing with $-\sigma$ instead of σ, we can see that $\text{g.l.b.}_{S\in\mathbf{A}_1^+}\sigma'(S)=$ $=\text{g.l.b.}_{S\in\mathbf{A}_1^+}\sigma''(S)=0$ as well. This means that $\sigma'=\sigma''=0$, i.e. $\sigma=0$, hence $\sigma_1=\sigma_2$. [7]) Thus we obtain

[5]) For two projections P and Q, $P\cap Q$ denotes the projection of $\mathfrak{H}$ onto the subspace $(P\mathfrak{H})\cap$ $\cap(Q\mathfrak{H})$.

[6]) J' also runs over all possible finite systems of elements of G.

[7]) For $\sigma_1, \sigma_2\in\mathcal{R}^+(\mathbf{A}, \mathcal{G})$, this fact has been already established in [3].

Proposition 1. *Each element of $\mathscr{R}(\mathbf{A}, \mathscr{G})$ is uniquely determined by its restriction to $\mathbf{A}^{\mathscr{G}}$.*

Suppose now that σ belongs to $\mathscr{R}^r(\mathbf{A}, \mathscr{G})$, and denote by F the projection in $\mathbf{A}^{\mathscr{G}}$ such that $\sigma(F) = \underset{T \in \mathbf{A}_I^+}{\text{l.u.b.}}\, \sigma(T)$ (cf. Theorem). Put $\sigma_1(T) = \sigma(FT)$ and $\sigma_2(T) = -\sigma((I-F)T)$ $(T \in \mathbf{A})$. Then, according to [4], σ_1 and σ_2 are ultra-weakly continuous positive linear forms on $\mathbf{A}$ with disjoint supports (cf. [1, chap. I, § 4, no. 6]) such that $\sigma = \sigma_1 - \sigma_2$. Moreover, it is evident that $\sigma_1, \sigma_2 \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$. Thus we have

Proposition 2. *Each element of $\mathscr{R}^r(\mathbf{A}, \mathscr{G})$ can be represented as a difference of two elements of $\mathscr{R}^+(\mathbf{A}, \mathscr{G})$ with disjoint supports. Therefore, each element of $\mathscr{R}(\mathbf{A}, \mathscr{G})$ is a finite linear combination of elements of $\mathscr{R}^+(\mathbf{A}, \mathscr{G})$. If $\mathbf{A}^{\mathscr{G}}$ reduces to the trivial von Neumann algebra of the scalars, then each element of $\mathscr{R}^r(\mathbf{A}, \mathscr{G})$ is either positive or negative* (i.e. a positive linear form multiplied by -1).

From this proposition we can conclude at once the following

Proposition 3. *$\mathbf{A}$ is $\mathscr{G}$-finite if and only if for every $T \in \mathbf{A}^+$, $T \neq 0$, there exists an element σ of $\mathscr{R}(\mathbf{A}, \mathscr{G})$ such that $\sigma(T) \neq 0$.*

In particular if $\mathscr{G}$ is the group of all inner automorphisms of $\mathbf{A}$, then $\mathscr{R}(\mathbf{A}, \mathscr{G})$ is identical with the space of all ultra-weakly continuous central (cf. [1, p. 275]) linear forms on $\mathbf{A}$. In this case Propositions 2 and 3 assert that

(i) *Each ultra-weakly continuous central linear form on $\mathbf{A}$ is a linear combination of finite normal traces* (cf. [1, chap. I. § 6, def. 1]). *In particular if $\mathbf{A}$ is a factor, and $\sigma \neq 0$ is an ultra-weakly continuous central linear form on $\mathbf{A}$, then $\mathbf{A}$ is finite and σ is a scalar multiple of the canonical trace of $\mathbf{A}$. Furthermore if $\mathbf{A}$ is a properly infinite von Neumann algebra* (cf. [1. chap. I, § 6. def. 5]) *then there exists no non-zero ultra-weakly continuous central linear form on $\mathbf{A}$.*

(ii) *$\mathbf{A}$ is finite if and only if for every $T \in \mathbf{A}^+$, $T \neq 0$ there exists an ultra-weakly continuous central linear form σ on $\mathbf{A}$ such that $\sigma(T) \neq 0$.*

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Über die Reflexivität gewisser Banachräume

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1. Wir bezeichnen mit M die Klasse der Folgen $A=\{a_n\}_1^\infty$, für die die Reihe

(1) $$\sum_{n=1}^{\infty} a_n \varphi_n(x)$$

bei jedem im Grundintervall (0, 1) orthonormierten System $\varphi=\{\varphi_n(x)\}$ fast überall konvergiert. Offensichtlich ist M mit den gewöhnlichen vektoriellen Operationen ein linearer Raum. Wir setzen

$$\|A\| = \lim_{N\to\infty} \left\{\sup_{\varphi} \int_0^1 \max_{1\leq i\leq j\leq N} (a_i\varphi_i(x)+\cdots+a_j\varphi_j(x))^2\,dx\right\}^{1/2},$$

wobei das Supremum für jedes in (0, 1) orthonormierte System φ gebildet ist.

In einer vorigen Arbeit [1] haben wir gezeigt, daß $A\in M$ mit $\|A\|<\infty$ äquivalent ist und daß M mit der Norm $\|A\|$ ein Banachraum ist.

In dieser Note werden wir Folgendes beweisen:

Satz. *Der Banachraum M ist reflexiv.*

2. Für eine Folge A bezeichne $A(i,j)$ $(1\leq i\leq j\leq\infty)$ die Folge $\underbrace{0,\cdots,0}_{i-1}, a_i, \cdots$ $\cdots, a_j, 0, \cdots$. Nach der Definition von $\|A\|$ gilt

(2) $$\|A\| = \lim_{N\to\infty} \|A(1,N)\|.$$

Durch einfache Rechnung erhalten wir

(3) $$\left\{\sum_{n=1}^{\infty} a_n^2\right\}^{1/2} \leq \|A\|,$$

(4) $$\|A(i,j)\|\leq\|A\| \qquad (1\leq i\leq j\leq\infty).$$

Weiterhin gilt

(5) $$\|A(N,\infty)\|\to 0 \qquad (N\to\infty;\ A\in M).$$

(Siehe [1], Satz IV.)

Wir bezeichnen mit E_n die Folge $\overbrace{0, \cdots, 0}^{n-1}, 1, 0, \cdots$ $(n=1, 2, \cdots)$. Es sei M^* der konjugierte Raum von M. Für $F \in M^*$ folgt aus (5)

$$\sum_{n=1}^{N} a_n F(E_n) = F(A(1, N)) \to F(A) \qquad (A \in M),$$

also gilt

(6) $$F(A) = \sum_{n=1}^{\infty} a_n f_n \qquad (A \in M)$$

mit $f_n = F(E_n)$. Für $F \in M^*$ und für eine natürliche Zahl N setzen wir

$$F_N(A) = \sum_{n=1}^{N} a_n f_n \qquad (A \in M).$$

Offensichtlich ist $F_N \in M^*$ $N=1, 2, \cdots)$.

3. Wir benötigen den folgenden

Hilfssatz. *Für* $F \in M^*$ *gilt* $\|F-F_N\| \to 0$ $(N \to \infty)$.

Beweis. Es sei N eine natürliche Zahl und ε eine beliebige positive Zahl. Auf Grund von (6) gibt es zu ε eine Folge $A \in M$ mit $\|A\| \leqq 1$ und

(7) $$\sum_{n=N+2}^{\infty} a_n f_n = (F-F_{N+1})(A) \geqq \|F-F_{N+1}\| - \varepsilon.$$

Auf Grund von (4) können wir $a_n = 0$ $(n=1, \ldots, N+1)$ annehmen. So ist

(8) $$\|F-F_N\| = \sup_{\|C\| \leqq 1} \sum_{n=N+1}^{\infty} c_n f_n \geqq \sum_{n=N+2}^{\infty} a_n f_n.$$

Aus (7) und (8) ergibt sich

(9) $$\|F-F_N\| \geqq \|F-F_{N+1}\| \qquad (N = 1, 2, \ldots).$$

Weiterhin gilt offensichtlich

(10) $$\|F_N\| \to \|F\| \qquad (N \to \infty; \quad F \in M^*).$$

Wir nehmen an, daß $\|F-F_N\| \not\to 0$. Auf Grund von (9) und (10) gibt es eine positive Zahl ϱ und eine Indexfolge $(0=)N(0) < \cdots < N(k) < \cdots$ mit $\|F_{N(k+1)} - F_{N(k)}\| > \varrho$ $(k=0, 1, \cdots)$. (Hier ist $F_0 = O$; d.h. es gilt $F_0(A) = 0$ überall in M.) So existieren Folgen $C(k)(\in M)(k=0, 1, \cdots)$ mit $\|C(k)\| \leqq 1$ und

(11) $$(F_{N(k+1)} - F_{N(k)})(C(k)) = \sum_{n=N(k)+1}^{N(k+1)} c_n(k) f_n > \varrho;$$

auf Grund von (4) können wir $c_n(k) = 0$ $(n=1, \cdots, N(k), N(k+1)+1, \cdots)$ annehmen.

Wir setzen $a_n = c_n(k)/(k+1)\log(k+2)$ $(N(k) < n \leqq N(k+1);\ k=0, 1, \cdots)$. Für ein beliebiges, in $(0, 1)$ orthonormiertes System φ bezeichnen wir die n-te

Partialsumme der Reihe (1) mit $s_n(x)$. Aus (3) folgt

$$\sum_{n=1}^{\infty} a_n^2 \leqq \sum_{k=0}^{\infty} \|C(k)\|^2/(k+1)^2 \log^2(k+2) < \infty. \tag{12}$$

Nach dem Riesz—Fischerschen Satz gibt es eine Funktion $f(x) \in L^2(0, 1)$ mit

$$\int_0^1 (s_n(x) - f(x))^2\, dx \to 0 \qquad (n \to \infty). \tag{13}$$

Für eine natürliche Zahl n sei k_0 derjenige Index, für welchen $N(k_0) < n \leqq N(k_0+1)$ gilt. Dann besteht offensichtlich

$$s_n^2(x) \leqq 3\big((s_n(x) - s_{N(k_0)}(x))^2 + (s_{N(k_0)}(x) - f(x))^2 + f^2(x)\big) \leqq$$

$$\leqq 3\left(\sum_{k=0}^{\infty} \max_{N(k) < n \leqq N(k+1)} (s_n(x) - s_{N(k)}(x))^2 + \sum_{k=0}^{\infty} (s_{N(k)}(x) - f(x))^2 + f^2(x)\right),$$

und so ist

$$\max_{1 \leqq i \leqq j \leqq N} (s_j(x) - s_{i-1}(x))^2 \leqq$$

$$\leqq 12\left(\sum_{k=0}^{\infty} \max_{N(k) < n \leqq N(k+1)} (s_n(x) - s_{N(k)}(x))^2 + \sum_{k=0}^{\infty} (s_{N(k)}(x) - f(x))^2 + f^2(x)\right).$$

Auf Grund von (12) und (13) ergibt sich daraus

$$\|A\| \leqq 12\left(\sum_{k=0}^{\infty} \|A(N(k)+1, N(k+1))\|^2 + \sum_{k=0}^{\infty} \sum_{n=N(k)+1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} a_n^2\right) \leqq$$

$$12\left(\sum_{k=0}^{\infty} \|C(k)\|^2/(k+1)^2 \log^2(k+2) + \sum_{k=0}^{\infty} \|C(k)\|^2/(k+1) \log^2(k+2) + \sum_{n=1}^{\infty} a_n^2\right) < \infty.$$

Also gilt $A \in M$. Aus (11) folgt aber

$$F_{N(k)}(A) = \sum_{\varkappa=0}^{k-1} \sum_{n=N(\varkappa)+1}^{N(\varkappa+1)} a_n f_n > \varrho \sum_{\varkappa=0}^{k-1} ((\varkappa+1)\log(\varkappa+2))^{-1} \to \infty \qquad (k \to \infty).$$

Damit haben wir den Hilfssatz bewiesen.

4. Beweis des Satzes. Aus der Definition von M, weiterhin aus (3) und (5) folgt, daß die Folge $\{E_n\}$ $(n = 1, 2, \cdots)$ eine Basis von M bildet. Auf Grund der Definition von M und (5) ist die folgende Bedingung erfüllt:

a) gilt für eine Folge A $\left\|\sum_{k=1}^{n} a_k E_k\right\| \leqq K(<\infty)$ $(n = 1, 2, \cdots)$, dann ist die Reihe $\sum_{k=1}^{\infty} a_k E_k$ in M konvergent. (Die Basis $\{E_n\}$ ist „boundedly complete".)

Weiterhin folgt aus dem Hilfssatz

b) $$\lim_{N\to\infty} \sup_{\substack{\|A\| \leqq 1 \\ A \in B_{N+1}}} |F(A)| = 0$$

für jedes lineare Funktional $F \in M^*$, wobei B_{N+1} den von den Elementen E_n $(n = N+1, \cdots)$ aufgespannten linearen Unterraum bezeichnet. (Die Basis $\{E_n\}$ ist „shrinking".) Aus a) und b), durch Anwendung eines Satzes von R. C. JAMES ([1], Satz I) ergibt sich die Reflexivität von M.

5. Es sei $\Lambda = \{\lambda_n\}$ $(n = 1, 2, \cdots)$ eine Folge von Zahlen mit $\lambda_n \geqq 1$, $\lambda_n \to \infty$ $(n \to \infty)$. Wir bezeichnen mit $M(\Lambda)$ die Klasse der Folgen A, für die die Reihe (1) bei jedem im Grundintervall (0, 1) orthonormierten System φ mit

$$\sup_{\nu} \int_0^1 \frac{1}{\lambda_{\nu(x)}} \left(\int_0^1 \left| \sum_{k=1}^{\nu(x)} \varphi_k(x) \varphi_k(t) \right| dt \right) dx \leqq 1 \tag{15}$$

fast überall konvergiert, wobei das Supremum für jede meßbare Funktion $\nu(x)$ mit natürlichen Zahlen als Werten gebildet wird. $M(\Lambda)$ ist mit den gewöhnlichen vektoriellen Operationen ein linearer Raum. Für eine Folge A setzen wir

$$\|A; \Lambda\| = \lim_{N \to \infty} \left(\sup_{\varphi} \int_0^1 \max_{1 \leqq i \leqq j \leqq N} |a_i \varphi_i(x) + \cdots + a_j \varphi_j(x)| \, dx \right),$$

wobei das Supremum für jedes in (0, 1) orthonormierte System φ mit (15) gebildet ist.

In den Arbeiten [2], [3] haben wir gezeigt, daß $A \in M(\Lambda)$ mit $\|A; \Lambda\| < \infty$ äquivalent ist und daß $M(\Lambda)$ mit der Norm $\|A; \Lambda\|$ ein Banachraum ist.

Ähnlicherweise kann man beweisen, daß $M(\Lambda)$ *reflexiv ist.* (2) und (4) gelten nämlich auch für die Norm $\|A; \Lambda\|$; das Analogon von (5) haben wir in der Arbeit [3] bewiesen. Weiterhin, mit einer in [4] angewandten Methode kann man zeigen, daß

$$\|A; \Lambda\| \geqq c \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2}$$

mit einer Konstante $c > 0$ besteht.

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Ein Divergenzsatz für Fourierreihen

Von KÁROLY TANDORI in Szeged

Herrn Profesor Georg Alexits zum 70. Geburtstag gewidmet

1. Für ein ε, $0 \leqq \varepsilon < 1$, ist $\Phi_\varepsilon(x) = x(\log\log(x+e^e))^\varepsilon$ eine in $[0, \infty)$ streng wachsende, nach unten konvexe Funktion, $\Phi_\varepsilon(0) = 0$. Mit L_{Φ_ε} bezeichnen wir die Klasse der 2π-periodischen, meßbaren Funktionen f mit endlichem $\|f\|_{\Phi_\varepsilon} = \int_0^{2\pi} \Phi_\varepsilon(|f(x)|)dx$; $\tilde{f}$ bedeutet die zu f trigonometrisch-konjugierte Funktion.

V. I. PROCHORENKO [2] hat bewiesen, daß es eine Funktion f derart gibt, daß $f \in L_{\Phi_\varepsilon}$ für jedes ε, $0 \leqq \varepsilon < 1$, gilt, und die Fourierreihe von f fast überall divergiert.

Durch einfache Modifizierung eines Beispieles von KOLMOGOROFF [1] und mit Anwendung gewisser Ideen von STEIN [3], TAĬKOV [4] und YUNG-MING CHEN [5], werden wir die folgende, ziemlich schärfere Behauptung beweisen.

Satz. *Es gibt eine Funktion f derart, daß $f, \tilde{f} \in L_{\Phi_\varepsilon}$ für jedes ε gilt* $(0 \leqq \varepsilon < 1)$, *weiterhin die Fourierreihen von f und $\tilde{f}$ überall divergieren.*

Es seien $D_n(x)$ die n-te Dirichletsche und $K_n(x)$ die n-te Fejérsche Kernfunktion, ferner seien $D_n^*(x) = D_n(x) - \cos nx/2$. Bekanntlich hat man $D_n^*(x) = \sin nx \Big/ \left(2 \operatorname{tg} \frac{x}{2}\right)$

$$\int_{-\pi}^{\pi} K_n(x)\,dx = \pi, \qquad 0 \leqq K_n(x) \leqq 2n \; (-\infty < x < \infty), \qquad K_n(x) \geqq c_1 n \quad \left(|x| \leqq \frac{\pi}{4n}\right).$$

($c_1, c_2, \cdots$ bezeichnen positive Konstanten.) Weiterhin sei

$$V_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx + \sum_{k=n+1}^{2n} \left(1 - \frac{k-n}{n+1}\right) \cos kx.$$

Durch eine Abelsche Umformung ergibt sich:

$$V_n(x) = \frac{2n+1}{n+1} K_{2n}(x) - K_n(x),$$

folglich gilt:

$$|V_n(x)| \leqq c_2 n \quad (-\infty < x < \infty), \qquad \int_0^{2\pi} |V_n(x)|\, dx \leqq c_3.$$

3. Es sei $\{\lambda_n\}$ eine monoton ins Unendliche strebende Folge von positiven Zahlen, für die

(1) $$\lambda_n = O((\log n)^\varepsilon)$$

bei jedem ε, $0 < \varepsilon \leqq 1$, besteht. Für $n \geqq 8$ setzen wir

$$f_n(x) = \frac{\lambda_n}{n \log n} \cdot \left\{\sum_{k=0}^{[n/2]} K_{[n/2]^2}\left(x - \frac{\pi}{n}k\right) + (\cos 8 \cdot 2nx + \cos 2 \cdot 8 \cdot 2nx) \sum_{k=n+1}^{n+[n/2]} V_{8^k n}\left(x - \frac{2\pi}{n}(k-n)\right)\right\}.$$

($[n/2]$ bezeichnet den ganzen Teil von $n/2$.) Nach den obigen gilt

(2) $$|f_n(x)| \leqq c_4 8^{2n},$$

(3) $$\int_0^{2\pi} |f_n(x)|\, dx \leqq c_5 \lambda_n / \log n.$$

Die m-te Partialsumme der Fourierreihe einer Funktion g bezeichnen wir mit $s_m(g; x)$. Es sei

$$G_n = \bigcup_{k=0}^{[n/2]} \left[\frac{\pi}{n}k - \frac{\pi}{n^2}, \frac{\pi}{n}k + \frac{\pi}{n^2}\right].$$

Offensichtlich ist

(4) $$s_{[n/2]^2}(f_n; x) \geqq c_6 \lambda_n \quad (x \in G_n).$$

Wir setzen

$$g_n(x) = \frac{\lambda_n}{n \log n} \sum_{k=n+1}^{n+[n/2]} V_{8^k n}\left(x - \frac{2\pi}{n}(k-n)\right).$$

Es sei $1 \leqq k_0 \leqq [n/4]$ eine ganze Zahl und $\frac{2\pi}{n}(k_0 - 1) < x < \frac{2\pi}{2} k_0$. Für positive ganze Zahlen a und m mit

(5) $$2 \cdot 8^{k_0+n-1} n < mn - an < mn + an < 8^{k_0+n} n$$

erhalten wir durch eine einfache Rechnung

(6) $$s_{mn+an}(g; x) - s_{mn-an-1}(g; x) =$$

$$= \frac{\lambda_n}{n \log n} \sum_{k=n+k_0}^{n+[n/2]} \sum_{l=mn-an}^{mn+an} \cos l\left(x - \frac{2\pi}{n}(k-n)\right) =$$

$$= \frac{\lambda_n}{2n \log n} \sum_{k=n+k_0}^{n+[n/2]} \cos mn\left(x - \frac{2\pi}{n}(k-n)\right) D_{an}\left(x - \frac{2\pi}{n}(k-n)\right) =$$

$$= \frac{\lambda_n \cos mnx}{2n \log n} \sum_{k=n+k_0}^{n+[n/2]} D_{an}^*\left(x - \frac{2\pi}{n}(k-n)\right) +$$

$$+ \frac{\lambda_n \cos mnx}{4n \log n} \sum_{k=n+k_0}^{n+[n/2]} \cos an\left(x - \frac{2\pi}{n}(k-n)\right) =$$

$$= \frac{\lambda_n \cos mnx \sin anx}{2n \log n} \sum_{k=n+k_0}^{n+[n/2]} \left(\operatorname{tg} \frac{x-2\pi(k-n)/n}{2}\right)^{-1} +$$

$$+ \frac{\lambda_n \cos mnx}{4n \log n} \sum_{k=n+k_0}^{n+[n/2]} \cos anx = S_1(x) + S_2(x).$$

Offensichtlich gilt

(7) $$|S_2(x)| \leqq c_7 \quad (-\infty < x < \infty).$$

Für $n+k_0 \leqq k \leqq n+[n/2]$ ist $-\pi < x - 2\pi(n-k)/n < 0$, und so, auf Grund der Ungleichung $|\operatorname{tg} x| \leqq \sqrt{2}|x|$ $(|x| \leqq \pi/4)$, ergibt sich

(8) $$|S_1(x)| \geqq \frac{\lambda_n |\cos mnx \cdot \sin anx|}{2n \log n} \sum_{k=n+k_0}^{n+[n/4]}{}' \left|\operatorname{tg} \frac{x-2\pi(n-k)/n}{2}\right|^{-1} \geqq$$

$$\geqq c_8 \lambda_n |\cos mnx \cdot \sin anx|.$$

Wir beweisen, daß

(9) $$\max |\cos m\,nx \sin anx| \geqq c_9$$

für $x \in \left[\frac{2\pi}{n}(k_0-1), \frac{2\pi}{n}k_0\right] - G_n$ besteht, wobei das Maximum für positive Zahlen a, m mit (5) gebildet ist. Um (9) zu beweisen, nehmen wir $\frac{2\pi}{n}(k_0-1) + \frac{\pi}{n^2} < x < \frac{2\pi}{n}(k_0-1) + \frac{\pi}{n} - \frac{\pi}{n^2}$ an. Ist

(10) $$\frac{2\pi}{n}(k_0-1) + s\frac{\pi}{n^2} \leqq x < \frac{2\pi}{n}(k_0-1) + (s+1)\frac{\pi}{n^2}$$

mit $\frac{n}{4} \leqq s \leqq \frac{3n}{4} - 1$, dann gilt $\sin\frac{\pi}{4} \leqq \sin nx$. Es sei $m_0 = (2.\ 8^{k_0+n-1}n + 2n)/n$. Für $m = m_0$, $a = 1$, bzw. für $m = 2m_0$, $a = 1$ ist (5) offensichtlich erfüllt; weiterhin gilt $\max(|\cos mnx|, |\cos 2mnx|) \geqq c_{10}$ überall. So ist $\max(|\cos m_0\, nx \cdot \sin anx|, \cos 2m_0 nx \cdot \sin anx|) \geqq c_{11}$, woraus (9) im Falle (10) mit $n/4 \leqq s \leqq 3n/4 - 1$ folgt. Gilt aber (10) mit $1 \leqq s < n/4$, so gibt es eine positive ganze Zahl c mit $n/4 \leqq 2cs \leqq \leqq \frac{3n}{4} - 1$ und $2c \leqq n$. Dann ist $\sin\frac{4}{\pi} \leqq \sin 2cnx$. Es sei nun $m_0 = (2.8^{k_0+n-1}n + 4cn)/n$.

Für $m=m_0, a=2c$, bzw. für $m=2m_0$, $a=2c$ ist (5) offensichtlich erfüllt, woraus (9) im Falle (10) mit $1\leqq s<n/4$ folgt. Da

$$\left|\sin an\left(\frac{2\pi}{n}(k_0-1)+\frac{\pi}{2n}-x\right)\right| = \left|\sin an\left(\frac{2\pi}{n}(k_0-1)+\frac{\pi}{2n}+x\right)\right| \qquad (a=1, a=2c)$$

besteht, gilt also (9) überall in dem Intervall

$$\left[\frac{2\pi}{n}(k_0-1)+\frac{\pi}{n^2}, \frac{2\pi}{n}(k_0-1)+\frac{\pi}{n}-\frac{\pi}{n^2}\right].$$

Im Falle $\frac{2\pi}{n}(k_0-1)+\frac{\pi}{n}+\frac{\pi}{n^2} < x < \frac{2\pi}{n}k_0-\frac{\pi}{n^2}$ folgt (9) aus

$$\left|\sin an\left(\frac{2\pi}{n}(k_0-1)+\frac{\pi}{n}-x\right)\right| = \left|\sin an\left(\frac{2\pi}{n}(k_0-1)+\frac{\pi}{n}+x\right)\right|.$$

Aus der Definition von $f_n(x)$ und aus (2), (6), (7), (8), (9) folgt

(11) $$\max_m |s_m(f_n; x)| \geqq c_{12}\lambda_n \qquad (x\in[0, \pi/4]).$$

Wir setzen

$$P_n(x) = (\cos 8^{3^n}x+\cos 2\cdot 8^{3^n}x+\sin 3\cdot 8^{3^n}x+\sin 6\cdot 8^{3^n}x)f_n(x).$$

$P_n(x)$ ist ein trigonometrisches Polynom:

(12) $$P_n(x) = \sum_{k=8^{3^n}-4\cdot 8^{2^n}}^{6\cdot 8^{3^n}+4\cdot 8^{2^n}} (a_k\cos kx+b_k\sin kx).$$

Offensichtlich gilt

$$\tilde{P}_n(x) = (\sin 8^{3^n}x+\sin 2\cdot 8^{3^n}x-\cos 3\cdot 8^{3^n}x-\cos 6\cdot 8^{3^n}x)f_n(x).$$

Für ein ε $(0\leqq\varepsilon<1)$ aus (1), (2), (3) und (11) folgt also

(13) $$\|P_n\|_{\Phi_\varepsilon} \leqq c_{13}, \quad \|\tilde{P}_n\|_{\Phi_\varepsilon} \leqq c_{13} \qquad (n=8, 9, \ldots),$$

(14) $$\max_m |s_m(P_n; x)| \geqq c_{14}\lambda_n, \quad \max_m |s_m(\tilde{P}_n; x)| \geqq c_{14}\lambda_n \qquad (x\in[0, \pi/4]).$$

4. Es sei $(8\leqq)n_1<\cdots<n_k<\cdots$ eine Indexfolge mit $\lambda_{n_k}\geqq k^3$ $(k=1, 2, \cdots)$. Mit Rademacherschen Funktionen $r_k(t)=\operatorname{sign}\sin 2^n\pi x$ bilden wir die Reihen

(15) $$\sum_{k=1}^{\infty}\frac{1}{k^3}r_k(t)P_{n_k}\left(x+\frac{\pi}{4}k\right),$$

(16) $$\sum_{k=1}^{\infty}\frac{1}{k^3}r_k(t)\tilde{P}_{n_k}\left(x+\frac{\pi}{4}k\right),$$

die ν-ten Partialsummen dieser Reihen bezeichnen wir mit $R_\nu(x, t)$, bzw. mit $\tilde{R}_\nu(x, t)$.

Aus (13) ergeben sich

$$\sum_{k=1}^{\infty}\int_0^{2\pi}\int_0^1 \Phi_\varepsilon\left(\frac{1}{k^3}\left|r_k(t)P_{n_k}\left(x+\frac{\pi}{4}k\right)\right|\right)dx\,dt \leqq c_{13}\sum_{k=1}^{\infty}\frac{1}{k^3}<\infty,$$

$$\sum_{k=1}^{\infty}\int_0^{2\pi}\int_0^1 \Phi_\varepsilon\left(\frac{1}{k^3}\left|r_k(t)\tilde{P}_{n_k}\left(x+\frac{\pi}{4}k\right)\right|\right)dx\,dt \leqq c_{13}\sum_{k=1}^{\infty}\frac{1}{k^3}<\infty,$$

woraus, wegen $\Phi_\varepsilon(x)\geqq x$, erhalten wir, daß die Reihen (15), (16) bei fast jedem t fast überall zu einer Funktion $f_t(x)$, bzw. $\tilde{f}_t(x)$ konvergieren. Da $\psi_\varepsilon(x)=\Phi_\varepsilon(\sqrt{x})$ eine nach unten konkave Funktion ist, ergibt sich durch Anwendung der Jensenschen Ungleichung

$$\int_0^{2\pi}\int_0^1 \Phi_\varepsilon(|R_\mu(x,t)-R_\nu(x,t)|)\,dx\,dt = \int_0^{2\pi}\left(\int_0^1 \Phi_\varepsilon(|R_\mu(x,t)-R_\nu(x,t)|)\,dt\right)dx =$$

$$=\int_0^{2\pi}\left(\int_0^1 \psi_\varepsilon(|R_\mu(x,t)-R_\nu(x,t)|^2)\,dt\right)dx \leqq \int_0^{2\pi}\psi_\varepsilon\left(\int_0^1 (R_\mu(x,t)-R_\nu(x,t))^2\,dt\right)dx =$$

$$=\int_0^{2\pi}\psi_\varepsilon\left(\sum_{k=\nu+1}^{\mu}\frac{1}{k^6}P_{n_k}^2\left(x+\frac{\pi}{4}k\right)\right)dx \leqq \sum_{k=\nu+1}^{\mu}\int_0^{2\pi}\psi_\varepsilon\left(\frac{1}{k^6}P_{n_k}^2\left(x+\frac{\pi}{4}k\right)\right)dx =$$

$$=\sum_{k=\nu+1}^{\mu}\int_0^{2\pi}\Phi_\varepsilon\left(\frac{1}{k^3}\left|P_{n_k}\left(x+\frac{\pi}{4}k\right)\right|\right)dx \leqq c_{13}\sum_{k=\nu+1}^{\mu}\frac{1}{k^3}\leqq c_{15}\frac{1}{\nu^2}\quad(\nu<\mu).$$

Für $\mu\to\infty$ erhalten wir

$$\int_0^{2\pi}\int_0^1 \Phi_\varepsilon(|f_t(x)-R_\nu(x,t)|)\,dx\,dt \leqq c_{15}\frac{1}{\nu^2}\quad(\nu=1,2,\ldots).$$

Daraus folgt

$$\int_0^{2\pi}\Phi_\varepsilon(|f_t(x)-R_\nu(x,t)|)\,dx\to 0\qquad(\nu\to\infty)$$

bei fast jedem t. Wegen $\Phi_\varepsilon(2x)\leqq c_{16}\Phi_\varepsilon(x)$ ist $R_\nu(x,t)\in L_{\Phi_\varepsilon}$, und so gilt $f_t(x)\in L_{\Phi_\varepsilon}$ bei fast jedem t. Da $\Phi_\varepsilon(x)\geqq x$ ist, gilt auch

$$\int_0^{2\pi}|f_t(x)-R_\nu(x,t)|\,dx\to 0\qquad(\nu\to\infty)$$

bei fast jedem t. Ähnliche Behauptungen können wir auch für die Reihe (16) beweisen.

Es gibt also eine Zahl t_0 derart, daß die Reihen (15) und (16) in der Norm von $L(0,2\pi)$ gegen den Funktionen $f_{t_0}(x)$ bzw. $\tilde{f}_{t_0}(x)$ konvergieren; weiterhin gelten $f_{t_0}(x), \tilde{f}_{t_0}(x)\in L_{\Phi_\varepsilon}$; wir können auch $r_k(t_0)\neq 0$ $(k=1,2,\cdots)$ annehmen. Aus (12) folgt, daß die Reihen (15) und (16) im Falle $t=t_0$ die Fourierreihen von

$f_{t_0}(x)$, bzw. von $\tilde{f}_{t_0}(x)$ sind, und $\tilde{f}_{t_0}(x)$ die konjugierte Funktion von $f_{t_0}(x)$ ist. Aus (14) erhalten wir weiterhin, daß für die Funktion $f(x)=f_{t_0}(x)$ auch die übrigen Forderungen des Satzes erfüllt sind.

5. Mit einer kleinen Modifizierung des obigen Beweises können wir auch die folgenden Behauptungen beweisen.

Es sei $0\leqq\varepsilon<1$ *und* $\{\mu_n\}$ *eine Folge von positiven Zahlen mit* $\mu_n=O((\log\log n)^{1-\varepsilon})$. *Dann gibt es eine Funktion* $f\in L_{\Phi_\varepsilon}$ *mit* $\tilde{f}\in L_{\Phi_\varepsilon}$ *derart, daß überall gilt:*

$$\overline{\lim_{n\to\infty}}\,|s_n(f;x)|/\mu_n=\overline{\lim_{n\to\infty}}\,|s_n(\tilde{f};x)|/\mu_n=\infty.$$

(Verschärfung eines Satzes von Yung-Ming Chen [5].)

Es sei $0\leqq\varepsilon<1$ *und* $\{\mu_n\}$ *eine Folge von positiven Zahlen mit* $\mu_n=O((\log n)^{1-\varepsilon})$. *Dann gibt es eine Funktion* $f\in L_{\Phi_\varepsilon}$ *mit* $\tilde{f}\in L_{\Phi_\varepsilon}$ *derart, daß überall gilt:*

$$\overline{\lim_{m,n\to\infty}}\,|s_m(f;x)-s_n(f;x)|/\mu_{|m-n|}=\overline{\lim_{m,n\to\infty}}\,|s_m(\tilde{f};x)-s_n(\tilde{f};x)|/\mu_{|m-n|}=\infty.$$

(Verschärfung eines Satzes von Stein [3].)

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On the T-summation of orthogonal series

By FERENC MÓRICZ in Szeged

Introduction

Let $\{\varphi_n(x)\}_1^\infty$ be an arbitrary orthonormal system (in abbreviation "ONS") in $[0, 1]$. We shall consider series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \tag{1}$$

with real coefficients, $\{c_n\} \in l^2$. By the Riesz—Fischer theorem, (1) converges in the mean to a square integrable function $f(x)$.

Let B be the class of those $\{c_n\}_1^\infty$ for which (1) converges almost everywhere (in abbreviation "a.e.") for every ONS in $[0, 1]$. (The set of divergence points may depend on the system $\{\varphi_n(x)\}$.) TANDORI [3] proved the following

Theorem. *For any sequence* $\mathfrak{c}=\{c_n\}_1^\infty$ *of real numbers set*

$$I(c_1, \dots, c_N) = \sup \int_0^1 \Big(\max_{1 \leqq i \leqq j \leqq N} |c_i \varphi_i(x) + \dots + c_j \varphi_j(x)| \Big)^2 \, dx,$$

the supremum being taken over all ONS in $[0, 1]$; *furthermore, define*

$$\|\mathfrak{c}\| = \lim_{N \to \infty} I^{1/2}(c_1, \dots, c_N) \qquad (\leqq \infty).$$

We have $\mathfrak{c} \in B$ *if and only if* $\|\mathfrak{c}\| < \infty$. *B is a Banach space with respect to the usual vector operations and the norm* $\|\mathfrak{c}\|$.

The aim of this paper is to extend this result to T-summability instead of convergence. More exactly, let $T=(a_{ik})_{i,k=1}^\infty$ be a double infinite matrix of real numbers satisfying the conditions

$$\lim_{i \to \infty} a_{ik} = 0 \qquad (k = 1, 2, \dots), \tag{2}$$

$$\lim_{i \to \infty} \sum_{k=1}^{\infty} a_{ik} = 1, \tag{3}$$

and

$$\sum_{k=1}^{\infty} |a_{ik}| \leq K \qquad (i = 1, 2, \ldots),\ ^{1)} \tag{4}$$

where K is a positive constant. In the sequel, we use $K, K_1, K_2, \ldots$ to denote positive constants. Set

$$A_{in} = \sum_{k=n}^{\infty} a_{ik} \qquad (n = 1, 2, \ldots).$$

We denote by $s_k(x)$ the kth partial sum of (1). The series (1) is called *T-summable* at the point $x \in [0, 1]$ if

$$t_i(x) = \sum_{k=1}^{\infty} a_{ik} s_k(x) = \sum_{n=1}^{\infty} A_{in} c_n \varphi_n(x)$$

exists for all i, and

$$\lim_{i \to \infty} t_i(x) = f(x).$$

Let $B(T)$ be the class of those $\{c_n\}_1^{\infty}$ for which (1) is T-summable a.e. for every ONS in [0, 1]. (The set, in the points of which (1) is not T-summable, depends on the system $\{\varphi_n(x)\}$.) We note that if (1) is T-summable a.e. for every ONS in [0, 1], then necessarily $\{c_n\} \in l^2$. For example, the Rademacher series $\sum c_n r_n(x)$ is not T-summable when $\sum c_n^2 = \infty$. (See ZYGMUND [5].) Hence we infer that $B(T) \subseteqq l^2$.

Our principal result is the following

Theorem 1. *Let T be a matrix satisfying conditions* (2), (3) *and* (4). *For any sequence $\mathfrak{c} = \{c_n\}_1^{\infty}$ of real numbers set*

$$I(T, \mathfrak{c}, N) = \sup \int_0^1 (\max_{1 \leq i \leq N} |t_i(x)|)^2 \, dx,\ ^{2)}$$

the supremum being taken over all ONS *in* [0, 1]; *furthermore, define*

$$\|\mathfrak{c}\|_T = \lim_{N \to \infty} I^{1/2}(T, \mathfrak{c}, N) \qquad (\leqq \infty). \tag{5}$$

We have $\mathfrak{c} \in B(T)$ if and only if $\|\mathfrak{c}\|_T < \infty$. $B(T)$ is a Banach space with respect to the usual vector operations and the norm $\|\mathfrak{c}\|_T$.

In a number of important special cases such as (C, α)-summability or $(R, \lambda_n, 1)$-summability (see ALEXITS [1], p. 139) there exists an increasing sequence $\mathfrak{n} = \{n_i\}$

[1]) We note that the conditions (2)—(4) are necessary and sufficient for the permanence of the T-summation. (See ZYGMUND [4], p. 74.)

[2]) This is evidently a non-decreasing function of N.

of natural numbers such that, under $\mathfrak{c} \in l^2$, the a.e. T-summability of (1) for every ONS is equivalent to the a.e. convergence of the sequence of the n_ith partial sums of (1). In this special case, we have $B(T)=B(T_{\mathfrak{n}})$, where $T_{\mathfrak{n}}$ is defined as follows: for every i put $a_{i,n_i}=1$ and $a_{ik}=0$ if $k \neq n_i$; then our Theorem 1 includes Theorem II of TANDORI [3] as a particular case. We note that, as MENCHOFF [2] showed, there exists a matrix T with (2), (3) and (4) such that for any increasing sequence $\mathfrak{n}$ of natural numbers we have $B(T) \neq B(T_{\mathfrak{n}})$.

The following theorems are the extensions of those of TANDORI that can also be found in his cited paper.

We say that (1) is "*boundedly*" T-summable if

(i) it is T-summable a.e. in [0, 1];

(ii) the T-means $t_i(x)$ are majorized by some square integrable function, the square integral of which has a bound depending only on the sequence $\mathfrak{c}$ of coefficients.

Theorem 2. *The a.e. T-summability of the series* (1) *for every* ONS *is equivalent to its bounded T-summability for every* ONS *in* [0, 1].

The following three theorems contain assertions concerning some properties of the norm $\|\mathfrak{c}\|_T$ and of the class $B(T)$.

Theorem 3. *Let* $\mathfrak{c}=\{c_n\}_1^\infty$ *and* $\mathfrak{d}=\{d_n\}_1^\infty$ *be two sequences of real numbers with* $|c_n| \leqq |d_n|$ $(n=1, 2, \cdots)$. *If* $\mathfrak{d} \in B(T)$ *then* $\mathfrak{c} \in B(T)$ *and* $\|\mathfrak{c}\|_T \leqq \|\mathfrak{d}\|_T$.

Theorem 4. *Let* $\mathfrak{c}_m=\{c_{mn}\}_{n=1}^\infty$ $(m=1, 2, \cdots)$ *be such that, for every fixed* n, c_{mn} *is a decreasing sequence in m and tends to* 0. *Suppose, moreover, that* $\|\mathfrak{c}_1\|_T < \infty$. *Then* $\|\mathfrak{c}_m\|_T \to 0$ $(m \to \infty)$.

Theorem 5. $B(T)$ *is separable.*

Finally we note, without any proof, that Theorem 1 remains valid if (5) is replaced by

$$\|\mathfrak{c}\|_T^{(p)} = \lim_{N \to \infty} I_p^{1/p}(T, \mathfrak{c}, N) \qquad (1 \leqq p \leqq 2),$$

where

$$I_p(T, \mathfrak{c}, N) = \sup \int_0^1 \left(\max_{1 \leqq i \leqq N} |t_i(x)| \right)^p dx,$$

the supremum being taken over all ONS in [0,1].

§1. Lemmas

The proofs of the theorems depend on several lemmas. First, let us introduce the quantity

$$J(\mathfrak{c}, M, N) = J(T, \mathfrak{c}, M, N) = \sup \int_0^1 \Big(\max_{M \leqq i < j \leqq N} |t_j(x) - t_i(x)| \Big)^2 dx =$$

$$= \sup \int_0^1 \left(\max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\infty} (A_{jn} - A_{in}) c_n \varphi_n(x) \right| \right)^2 dx,$$

where M and N denote natural numbers with $M < N$, and the supremum is taken over all ONS in $[0, 1]$. Sometimes, if it does not cause any misunderstanding, instead of $I(T, \mathfrak{c}, N)$, $\|\mathfrak{c}\|_T$ and $B(T)$ we shall write $I(\mathfrak{c}, N)$, $\|\mathfrak{c}\|$ and B, respectively. It is obvious that

$$\frac{1}{2} I(\mathfrak{c}, N) - I(\mathfrak{c}, M) \leqq J(\mathfrak{c}, M, N) \leqq 4 I(\mathfrak{c}, N). \tag{6}$$

In the sequel we shall work with the projections P_μ, P^ν and P_μ^ν defined as follows: for any given $\mathfrak{c} = \{c_n\}_1^\infty$ we denote by $P_\mu \mathfrak{c}$ the sequence that comes from $\mathfrak{c}$ by replacing the first $\mu - 1$ components of $\mathfrak{c}$ with 0, that is $P_\mu \mathfrak{c} = \{0, \cdots, 0, c_\mu, c_{\mu+1}, \cdots\}$; similarly, $P^\nu \mathfrak{c} = \{c_1, \cdots, c_\nu, 0, 0, \cdots\}$; and $P_\mu^\nu \mathfrak{c} = P_\mu P^\nu \mathfrak{c} = \{0, \cdots, 0, c_\mu, c_{\mu+1}, \cdots, c_\nu, 0, 0, \cdots\}$ $(1 \leqq \mu \leqq \nu)$.

In the following lemmas we always suppose that $\mathfrak{c} \in l^2$.

Lemma 1. *Let ε be a positive real number. Then there exists a natural number $N_0 = N_0(\varepsilon)$ such that*

$$I(\mathfrak{c}, N) \geqq (1 - \varepsilon) \sum_{n=1}^{\infty} c_n^2 - \varepsilon \tag{7}$$

holds for every $N \geqq N_0$; furthermore, for every natural number N and ν, we have

$$I(T^\nu \mathfrak{c}, N) \leqq K^2 \left(\sum_{n=1}^{\nu} |c_n| \right)^2. \tag{8}$$

Proof. To prove (7) we start with the relations

$$I(\mathfrak{c}, N) \geqq \int_0^1 t_N^2(x)\, dx = \sum_{n=1}^{\infty} A_{N,n}^2 c_n^2. \tag{9}$$

Because $\mathfrak{c} \in l^2$ we can fix the natural number $\nu_0 = \nu_0(\varepsilon)$ such that

$$\sum_{n=\nu_0+1}^{\infty} c_n^2 < \varepsilon.$$

By virtue of (2) and (3), there exists a natural number $N_0=N_0(\varepsilon)$ such that for every $l\geqq\nu_0$ we have

$$|A_{Nl}^2-1|\leqq 2\left|\left(\sum_{k=1}^{\infty}a_{Nk}\right)^2-1\right|+2\left(\sum_{k=1}^{l}a_{Nk}\right)^2\leqq\varepsilon \quad\text{if}\quad N\geqq N_0.$$

By (9) we get

$$I(\mathfrak{c},N)\geqq(1-\varepsilon)\sum_{n=1}^{\nu_0}c_n^2\geqq(1-\varepsilon)\left(\sum_{n=1}^{\infty}c_n^2-\varepsilon\right)\quad\text{if}\quad N\geqq N_0.$$

As to (8), it is sufficient to consider the following inequality:

$$\max_{1\leqq i\leqq N}\left|\sum_{n=1}^{\nu}A_{in}c_n\varphi_n(x)\right|\leqq K^2\sum_{n=1}^{\nu}|c_n\varphi_n(x)|.$$

Here we took (4) into consideration.

The proof of Lemma 1 is complete.

Lemma 2. *Let λ, μ, ν and N be natural numbers, $\lambda<\mu<\nu\leqq\infty$. Then we have*

$$I(P_\lambda^\mu\mathfrak{c},N)+I(P_{\mu+1}^\nu\mathfrak{c},N)\leqq I(P_\lambda^\nu\mathfrak{c},N),$$

and in particular

$$I(P^\nu\mathfrak{c},N)\leqq I(\mathfrak{c},N).$$

The proof of Lemma 2 is analogous to that of Lemma IV of Tandori [3].

Lemma 3. *Let M and N be natural numbers, $M<N$, and let ε be a positive real number. Then there exists a natural number $\nu_0=\nu_0(M,N,\varepsilon)$ such that*

(10) $$J(P_{\nu+1}\mathfrak{c},M,N)\leqq\varepsilon,$$

and

$$J(P^\nu\mathfrak{c},M,N)\geqq J(\mathfrak{c},M,N)-\varepsilon$$

hold for every $\nu\geqq\nu_0$. The similar assertions concerning $I(\mathfrak{c},N)$ are also valid.

Proof. It is sufficient to prove (10), as the second inequality is a simple consequence of (10), e.g. using the inequality $(a+b)^2\geqq a^2-2|a|\,|b|$. Let us consider an arbitrary ONS $\{\varphi_n(x)\}$ in [0, 1]. It is clear that

$$\left(\max_{M\leqq i<j\leqq N}\left|\sum_{n=\nu+1}^{\infty}(A_{jn}-A_{in})c_n\varphi_n(x)\right|\right)^2\leqq 4\sum_{i=M}^{N}\left(\sum_{n=\nu+1}^{\infty}A_{in}c_n\varphi_n(x)\right)^2.$$

Integrating over [0, 1] term by term, on account of (4) we get

$$\int_0^1\left(\max_{M\leqq i<j\leqq N}\left|\sum_{n=\nu+1}^{\infty}(A_{jn}-A_{in})c_n\varphi_n(x)\right|\right)^2dx\leqq 4(N-M+1)K^2\sum_{n=\nu+1}^{\infty}c_n^2<\varepsilon,$$

if ν is large enough, since $\mathfrak{c}\in l^2$. Since this estimate is valid for every ONS in [0, 1], we obtain (10).

This finishes the proof of Lemma 3.

Lemma 4. *Let μ be a natural number and let ε be a positive real number. Then there exists $M_0 = M_0(\mu, \varepsilon)$ such that*

$$J(P^{\mu}\mathfrak{c}, M, N) \leqq \varepsilon, \tag{11}$$

and

$$J(P_{\mu+1}\mathfrak{c}, M, N) \geqq J(\mathfrak{c}, M, N) - \varepsilon$$

hold whenever $M_0 \leqq M < N$.

Proof. It is also sufficient to prove (11). Let us consider an arbitrary ONS $\{\varphi_n(x)\}$ in [0, 1]. By a simple calculation we get

$$\int_0^1 \left(\max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\mu} (A_{jn} - A_{in}) c_n \varphi_n(x) \right| \right)^2 dx \leqq$$

$$\leqq \left(\max_{M \leqq i < j \leqq N} |A_{jn} - A_{in}| \right)^2 \int_0^1 \left(\sum_{n=1}^{\mu} |c_n \varphi_n(x)| \right)^2 dx \leqq \left(\sup_{M \leqq i < j} |A_{jn} - A_{in}| \right)^2 \left(\sum_{n=1}^{\mu} |c_n| \right)^2.$$

By virtue of (2) and (3), there exists a natural number M_0 such that for every $n \leqq \mu$ we have

$$(A_{jn} - A_{in})^2 \leqq 4 \left\{ \left(\sum_{k=1}^{\infty} a_{jk} - 1 \right)^2 + \left(\sum_{k=1}^{\infty} a_{ik} - 1 \right)^2 + \left(\sum_{k=1}^{\mu} a_{jk} \right)^2 + \left(\sum_{k=1}^{\mu} a_{ik} \right)^2 \right\} \leqq \frac{\varepsilon}{\left(\sum_{n=1}^{\mu} |c_n| \right)^2}$$

whenever $M_0 \leqq i < j$, whence

$$\int_0^1 \left(\max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\mu} (A_{jn} - A_{in}) c_n \varphi_n(x) \right| \right)^2 dx \leqq \varepsilon \quad \text{if} \quad M_0 \leqq M < N.$$

Since this is valid for every ONS in [0, 1], (11) follows.

Thus the proof is complete.

Lemma 5. *The inequality*

$$I^{1/2}(\mathfrak{c} + \delta, N) \leqq I^{1/2}(\mathfrak{c}, N) + I^{1/2}(\delta, N)$$

holds.

Lemma 6. *Let L, M and N be natural numbers, $L < M < N$. Then the inequalities*

$$J^{1/2}(\mathfrak{c}, L, N) \leqq J^{1/2}(\mathfrak{c}, L, M) + J^{1/2}(\mathfrak{c}, M, N),$$

and

$$I^{1/2}(\mathfrak{c}, N) \leqq I^{1/2}(\mathfrak{c}, M) + J^{1/2}(\mathfrak{c}, M, N)$$

hold.

The proofs of Lemma 5 and Lemma 6 are similar to that of Lemma II of Tandori [3].

Lemma 7. *Let ν and N be natural numbers. Then $I(P^{\nu}\mathfrak{c}, N)$ is a continuous function of the coefficients c_n.*

Proof. This is an immediate consequence of Lemma 1 and Lemma 5.

Lemma 8. *Let M_1 and N_1 be natural numbers, $M_1 < N_1$, and let ε be a positive real number. Then there exists a natural number $M_0 = M_0(M_1, N_1, \varepsilon) > N_1$ such that*

$$J(\mathfrak{c}, M_1, N_1) + J(\mathfrak{c}, M_2, N_2) \leqq J(\mathfrak{c}, M_1, N_2) + \varepsilon,$$

whenever $M_0 \leqq M_2 < N_2$.

Proof. By virtue of Lemma 3 there exists a natural number $\nu_0 = \nu_0(M_1, N_1, \varepsilon)$ such that

$$J(P^{\nu_0}\mathfrak{c}, M_1, N_1) \geqq J(\mathfrak{c}, M_1, N_1) - \frac{\varepsilon}{4}.$$

According to Lemma 4 there exists a natural number $M_0 = M_0(\nu_0, \varepsilon) = M_0(M_1, N_1, \varepsilon)$ such that

$$J(P_{\nu_0+1}\mathfrak{c}, M_2, N_2) \geqq J(\mathfrak{c}, M_2, N_2) - \frac{\varepsilon}{4},$$

whenever $M_0 \leqq M_2 < N_2$. Thus there exist ONS $\{\varphi_n(x)\}_1^{\nu_0}$ and $\{\psi_n(x)\}_{\nu_0+1}^{\infty}$ in $[0, 1]$ for which

$$(12)\qquad \begin{aligned} &\int_0^1 \left(\max_{M_1 \leqq i < j \leqq N_1} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \varphi_n(x) \right| \right)^2 dx \geqq J(\mathfrak{c}, M_1, N_1) - \frac{\varepsilon}{2}, \\ &\int_0^1 \left(\max_{M_2 \leqq i < j \leqq N_2} \left| \sum_{n=\nu_0+1}^{\infty} (A_{jn} - A_{in}) c_n \psi_n(x) \right| \right)^2 dx \geqq J(\mathfrak{c}, M_2, N_2) - \frac{\varepsilon}{2}. \end{aligned}$$

Set, for $n = 1, 2, \ldots, \nu_0$,

$$\chi_n(x) = \sqrt{2}\, \varphi_n(2x) \quad \text{if} \quad 0 \leqq x \leqq \frac{1}{2}, \text{ and } \chi_n(x) = 0 \text{ otherwise;}$$

and, for $n = \nu_0 + 1, \nu_0 + 2, \ldots$,

$$\chi_n(x) = \sqrt{2}\, \psi_n(2x - 1) \quad \text{if} \quad \frac{1}{2} < x \leqq 1, \text{ and } \chi_n(x) = 0 \text{ otherwise.}$$

It is obvious that $\{\chi_n(x)\}_1^{\infty}$ is an ONS in $[0, 1]$, and it follows from (12) that

$$\begin{aligned} &J(\mathfrak{c}, M_1, N_1) + J(\mathfrak{c}, M_2, N_2) - \varepsilon \leqq \\ &\leqq \int_0^1 \left(\max_{M_1 \leqq i < j \leqq N_1} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \varphi_n(x) \right| \right)^2 dx + \\ &+ \int_0^1 \left(\max_{M_2 \leqq i < j \leqq N_2} \left| \sum_{n=\nu_0+1}^{\infty} (A_{jn} - A_{in}) c_n \psi_n(x) \right| \right)^2 dx = \end{aligned}$$

$$= 2\int_0^{1/2} \left(\max_{M_1 \leqq i < j \leqq N_1} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \varphi_n(2x) \right| \right)^2 dx +$$

$$+ 2\int_{1/2}^{1} \left(\max_{M_2 \leqq i < j \leqq N_2} \left| \sum_{n=\nu_0+1}^{\infty} (A_{jn} - A_{in}) c_n \psi_n(2x-1) \right| \right)^2 dx \leqq$$

$$\leqq \int_0^1 \left(\max_{M_1 \leqq i < j \leqq N_2} \left| \sum_{n=1}^{\infty} (A_{jn} - A_{in}) c_n \chi_n(x) \right| \right)^2 dx \leqq J(\mathfrak{c}, M_1, N_2),$$

which concludes the proof.

Lemma 9. *Let* $\mathfrak{c}$ *and* $\mathfrak{d}$ *be such that* $|c_n| \leqq |d_n|$ $(n = 1, 2, \cdots)$. *Then for every* N *we have*

$$I(\mathfrak{c}, N) \leqq I(\mathfrak{d}, N).$$

The proof can be carried out exactly in the same way as that of Lemma V of TANDORI [3].

Lemma 10. *Let* $\mathfrak{c}$ *be such that* $\|\mathfrak{c}\| < \infty$. *Then there exists an increasing sequence* $\{N_r\}_0^\infty$ *of integers,* $N_0 = 1$, *with the following properties: for every* ONS $\{\varphi_n(x)\}$ *in* [0, 1] *we have*

$$\sum_{r=1}^{\infty} \int_0^1 (t_{N_r}(x) - f(x))^2 \, dx < \infty, \text{[3]} \tag{13}$$

and, moreover,

$$\sum_{r=1}^{\infty} J(\mathfrak{c}, N_{r-1}, N_r) < \infty. \tag{14}$$

Proof. First we shall choose an increasing sequence $\{i_k\}_1^\infty$ of natural numbers for which

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (A_{i_k,n} - 1)^2 c^{n2} < \infty. \tag{15}$$

$\|\mathfrak{c}\| < \infty$ implies, using Lemma 1, $\mathfrak{c} \in l^2$. Thus there exist two sequences $\nu_1 < \nu_2 < \cdots$ and $i_1 < i_2 < \cdots$ of natural numbers such that

$$\sum_{n=\nu_k+1}^{\infty} c_n^2 \leqq \frac{1}{2^k},$$

and for every $n \leqq \nu_k$, making use of (2) and (3),

$$|A_{i_k,n} - 1| \leqq \left| \sum_{l=1}^{\infty} a_{i_k,l} - 1 \right| + \sum_{l=1}^{\nu_k} |a_{i_k,l}| \leqq \frac{1}{2^k} \qquad (k = 1, 2, \ldots).$$

---

[3]) $f(x)$ admits an expansion convergent in the mean: $f(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$.

By (4) we get

$$\sum_{n=1}^{\infty} (A_{i_k,n}-1)^2 c_n^2 = \sum_{n=1}^{\nu_k} + \sum_{n=\nu_k+1}^{\infty} \leqq \frac{1}{2^{2k}} \sum_{n=1}^{\nu_k} c_n^2 + 4K^2 \sum_{n=\nu_k+1}^{\infty} c_n^2 \leqq$$

$$\leqq \frac{1}{2^k}\left(\sum_{n=1}^{\infty} c_n^2 + 4K^2\right),$$

whence (15) follows.

For the sake of brevity we write $J(M, N)$ instead of $J(\mathfrak{c}, M, N)$ in the remaining part of the proof. Set $N_0=1$ and $N_1=i_1$. By Lemma 8 we can select an index $N_2=i_{k_1}$ with $k_1>1$ such that

$$J(1, N_1)+J(k, l) \leqq J(1, l)+\tfrac{1}{2},$$

whenever $N_2 \leqq k < l$. In particular, replacing k by N_2 and l by $N_3=i_{k_1+1}$, we obtain

$$J(1, N_1)+J(N_2, N_3) \leqq J(1, N_3)+\tfrac{1}{2}.$$

Let us repeat the above argument. We get that there exists an index $N_4=i_{k_2}$ with $k_2>k_1+1$ such that

$$J(1, N_3)+J(k, l) \leqq J(1, l)+\tfrac{1}{4},$$

whenever $N_4 \leqq k < l$, and in particular

$$J(1, N_3)+J(N_4, N_5) \leqq J(1, N_5)+\tfrac{1}{4}$$

with $N_5=i_{k_2+1}$. Continuing this procedure we obtain an infinite sequence $N_1<N_2<\cdots$ of indices such that we have

$$J(1, N_{2r-1})+J(k, l) \leqq J(1, l)+\frac{1}{2^r}, \tag{16}$$

whenever $l>k \geqq N_{2r}=i_{k_r}$ $(i_{k_r}>i_{k_{r-1}+1})$, and in particular

$$J(1, N_{2r-1})+J(N_{2r}, N_{2r+1}) \leqq J(1, N_{2r+1})+\frac{1}{2^r}, \tag{17}$$

where $N_{2r+1}=i_{k_r+1}$.

Let ϱ be a natural number. Let us consider the inequalities (17) in turn for $r=1, 2, \cdots, \varrho$, and add them. Then we get

$$\sum_{r=1}^{\varrho} J(1, N_{2r-1})+\sum_{r=1}^{\varrho} J(N_{2r}, N_{2r+1}) \leqq \sum_{r=1}^{\varrho} J(1, N_{2r+1})+1,$$

whence

$$\sum_{r=0}^{\varrho} J(N_{2r}, N_{2r+1}) \leqq J(1, N_{2\varrho+1})+1 \qquad (\varrho = 1, 2, \ldots). \tag{18}$$

By (16), putting $k=N_{2r+1}$ and $l=N_{2r+2}$, we obtain

$$J(1, N_{2r-1})+J(N_{2r+1}, N_{2r+2}) \leqq J(1, N_{2r+2})+\frac{1}{2^r} \qquad (r = 1, 2, \ldots). \tag{19}$$

Let us consider the inequality (19) for every $r=1, 2, \cdots, \varrho$. By adding them, and using the fact that $J(1, N)$ is a non-decreasing function of N, we get

$$\sum_{r=1}^{\varrho} J(1, N_{2r+2})+1 \geqq \sum_{r=1}^{\varrho} J(1, N_{2r-1})+\sum_{r=1}^{\varrho} J(N_{2r+1}, N_{2r+2}) \geqq$$

$$\geqq \sum_{r=1}^{\varrho} J(1, N_{2r-2})+\sum_{r=1}^{\varrho} J(N_{2r+1}, N_{2r+2}),$$

therefore, we have

$$\sum_{r=1}^{\varrho} J(N_{2r-1}, N_{2r}) \leqq J(1, N_{2\varrho})+J(1, N_{2\varrho+2})+1. \tag{20}$$

Combining the results (18) and (20), we obtain

$$\sum_{r=1}^{2\varrho+1} J(N_{r-1}, N_r) \leqq 3J(1, N_{2\varrho+2})+2 \leqq 12I(\mathfrak{c}, N_{2\varrho+2})+2.$$

As $\|\mathfrak{c}\|<\infty$, we get (14). Since $\{N_r\}$ a subsequence of $\{i_k\}$, (13) is also satisfied.

The proof of Lemma 10 is complete.

Lemma 11. *Let M and N be natural numbers, $M<N$. There exists an* ONS $\{\psi_n(x)\}_1^\infty$ *of step functions in* $[0, 1]$ *and an interval* $E \subseteqq [0, \frac{1}{2}]$ *having the following properties:*

$$\max_{M \leqq i<j \leqq N} \left|\sum_{n=1}^{\infty} (A_{jn}-A_{in}) c_n \psi_n(x)\right| \geqq 2 \quad \textit{if} \quad x \in E,$$

and

$$|E| \geqq K_1 \min\left(\frac{1}{2}, J(\mathfrak{c}, M, N)\right).^{4)}$$

Proof. According to the definition of J there exists an ONS $\{\varphi_n(x)\}_1^\infty$ in $[0, 1]$ such that

$$\int_0^1 \left(\max_{M \leqq i<j \leqq N} \left|\sum_{n=1}^{\infty} (A_{jn}-A_{in}) c_n \varphi_n(x)\right|\right)^2 dx \geqq \frac{1}{2} J(\mathfrak{c}, M, N).$$

By virtue of Lemma 3 there exists a natural number ν_0 such that

$$\int_0^1 \left(\max_{M \leqq i<j \leqq N} \left|\sum_{n=1}^{\nu_0} (A_{jn}-A_{in}) c_n \varphi_n(x)\right|\right)^2 dx \geqq \frac{1}{4} J(\mathfrak{c}, M, N). \tag{21}$$

Let ε be an arbitrary positive real number, $\varepsilon<1$. We consider a system $\{\chi_n(x)\}_1^{\nu_0}$ of step functions for which

$$\int_0^1 (\varphi_n(x)-\chi_n(x))^2 dx \leqq \varepsilon^2 \qquad (n=1,2, \ldots, \nu_0).$$

---

4) $|E|$ denotes the Lebesgue measure of the set E.

Set

$$\alpha_{ln}=\int_0^1 \chi_l(x)\chi_n(x)\,dx \qquad (l, n=1, 2, \ldots, \nu_0),$$

and

$$\eta_n=\sum_{l=1}^{n-1}|\alpha_{ln}|+\sum_{l=n+1}^{\nu_0}|\alpha_{ln}| \qquad (n=1, 2, \ldots, \nu_0).$$

We get by a simple calculation that

$$\int_0^1\left(\max_{M\leqq i<j\leqq N}\left|\sum_{n=1}^{\nu_0}(A_{jn}-A_{in})c_n\chi_n(x)\right|\right)^2 dx \geqq \frac{1}{8}J(\mathfrak{c}, M, N), \tag{22}$$

and

$$\int_0^1\left(\max_{M\leqq i<j\leqq N}\left|\sum_{n=1}^{\nu_0}(A_{jn}-A_{in})c_n\left(1-\frac{1}{\sqrt{\alpha_{nn}+\eta_n}}\right)\chi_n(x)\right|\right)^2 dx \leqq \frac{1}{16}J(\mathfrak{c}, M, N), \tag{23}$$

provided ε is small enough [5]).

Now we continue $\chi_n(x)$ on $[0, 2]$ so that we divide $(1, 2]$ into as many equal parts as there exist pairs of numbers l, n with $1\leqq l, n\leqq \nu_0, l\neq n$. We denote the single subintervals by I_{ln}, and then define for $x\in(1, 2]$ the values of the function $\chi_n(x)$ $(n\leqq \nu_0)$ as follows:

$$\chi_n(x)=\begin{cases}\sqrt{\tfrac{1}{2}\nu_0(\nu_0-1)|\alpha_{ln}|} & x\in I_{nl},\\ -\sqrt{\tfrac{1}{2}\nu_0(\nu_0-1)|\alpha_{ln}|}\operatorname{sign}\alpha_{ln} & x\in I_{ln} \quad (l=1, 2, \ldots, \nu_0;\ l\neq n).\end{cases}$$

The functions $\chi_n(x)$ are orthogonal to each other in $[0, 2]$ since for $l\neq n$

$$\int_0^2 \chi_l(x)\chi_n(x)\,dx=\int_0^1+\int_1^2=\int_0^1+\int_{I_{ln}}+\int_{I_{nl}}=\alpha_{ln}-|\alpha_{ln}|\operatorname{sign}\alpha_{ln}=0.$$

Furthermore, we have

$$\int_0^2 \chi_n^2(x)\,dx=\int_0^1\chi_n^2(x)\,dx+\sum_{l=1}^{n-1}|\alpha_{ln}|+\sum_{l=n+1}^{\nu_0}|\alpha_{ln}|=\alpha_{nn}+\eta_n.$$

Setting

$$\bar{\chi}_n(x)=\frac{1}{\sqrt{\alpha_{nn}+\eta_n}}\chi_n(x),$$

we get an ONS of step functions in $[0, 2]$, and from (22) and (23) we obtain

$$\int_0^2\left(\max_{M\leqq i<j\leqq N}\left|\sum_{n=1}^{\nu_0}(A_{jn}-A_{in})c_n\bar{\chi}_n(x)\right|\right)^2 dx \geqq \frac{1}{32}J(\mathfrak{c}, M, N). \tag{24}$$

---

[5]) To show (22), we can, for example, use the inequality $(a+b)^2\geqq a^2-2|a||b|$, and to show (23), we make use of another inequality $\left|1-1/\sqrt{1+a}\right|\leqq|a|$ if $a>K_2$, where $-1<K_2<0$.

Let us consider the step function

$$S(x) = \max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \bar{\chi}_n(x) \right|.$$

We can divide [0, 2] into a finite number of subintervals $J_1, J_2, \dots, J_r$ such that $S(x)$ has a constant value w_ϱ on each subinterval J_ϱ ($\varrho = 1, 2, \dots, r$). Set

$$\sum_{\varrho=1}^{r} w_\varrho^2 |J_\varrho| = A.$$

Without loss of generality, we may assume that $A \leqq 2$. Putting

$$u_0 = 0, \quad u_\varrho = \frac{1}{4} \sum_{\sigma=1}^{\varrho} w_\sigma^2 |J_\sigma| \qquad (\varrho = 1, 2, \dots, r),$$

and

$$\bar{\varphi}_n(x) = \begin{cases} \dfrac{2}{w_{\varrho+1}} \bar{\chi}_n \left(\dfrac{4}{w_{\varrho+1}^2} (x - u_\varrho) + \sum_{\sigma=1}^{\varrho} |J_\sigma| \right) & \text{if } x \in [u_\varrho, u_{\varrho+1}) \\ & (w_\varrho \neq 0;\ \varrho = 0, 1, \dots, r-1), \\ 0 \quad \text{otherwise in } [0, 1], \end{cases}$$

we can see that $\{\bar{\varphi}_n(x)\}_1^{\nu_0}$ is an ONS in [0, 1]. Set $E = [0, u_r)$. It is clear that $E \subseteqq [0, \frac{1}{2}]$, and by virtue of (24)

$$|E| \geqq \min \left(\frac{1}{2}, \frac{1}{32} J(\mathfrak{c}, M, N) \right).$$

On account of the definition of the functions $\bar{\varphi}_n(x)$, we have

$$\max_{M \leqq i < j \leqq N} \left| \sum_{n=1}^{\nu_0} (A_{jn} - A_{in}) c_n \bar{\varphi}_n(x) \right| \geqq 2 \quad \text{if} \quad x \in E. \tag{25}$$

Since the functions $\bar{\varphi}_n(x)$ with $n \leqq \nu_0$ identically vanish outside $[0, \frac{1}{2}]$, we can give an ONS $\{\psi_n(x)\}_1^\infty$ of step functions in [0, 1] in a trivial manner such that we have $\psi_n(x) = \bar{\varphi}_n(x)$ for $n \leqq \nu_0$, and $\psi_n(x) = 0$ if $x \in [0, \frac{1}{2}]$ for every $n \geqq \nu_0 + 1$. This does not affect the inequality (25), and concludes the proof of Lemma 11.

§2. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. (A) *Sufficiency.* Assume that $\|\mathfrak{c}\| < \infty$. By virtue of Lemma 10 there exists an increasing sequence $\{N_r\}$ of natural numbers such that both (13) and (14) are convergent. Applying B. Levi's theorem, we get on the one hand that the subsequence $\{t_{N_r}(x)\}$ converges a.e., on the other hand that

$$\delta_r(x) = \max_{N_r \leqq i < j \leqq N_{r+1}} |t_j(x) - t_i(x)| \to 0 \qquad (r \to \infty).$$

It is obvious that for $N_{r-1} < n < N_r$

$$|t_n(x) - t_{N_r}(x)| \leqq \delta_r(x) \to 0 \qquad (r \to \infty),$$

and the proof of the sufficiency is complete.

In the course of this proof we have obtained the following result: *if there exists an increasing sequence $\{N_r\}$ of integers such that both the subsequence $\{t_{N_r}(x)\}$ is convergent a.e. and*

$$\sum_{r=1}^{\infty} J(\mathfrak{c}, N_{r-1}, N_r) < \infty$$

holds, then the series (1) *is T-summable a.e.*

(B) *Necessity.* Suppose $\|\mathfrak{c}\| = \infty$. Using Lemma 6, we get that for any fixed natural number M

(26) $$\lim_{N \to \infty} J(\mathfrak{c}, M, N) = \infty$$

holds. We shall define by induction two sequences $1 = M_1 < N_1 < M_2 < N_2 < \cdots$ and $0 = \nu_1 < \nu_2 < \cdots$ of integers, depending only on T and $\mathfrak{c}$, such that

(27) $$J(\mathfrak{c}, M_r, N_r) \geqq 1 \qquad (r = 1, 2, \ldots),$$

(28) $$\sum_{r=1}^{\infty} J(P^{\nu_r}\mathfrak{c}, M_r, N_r) < \infty,$$

and

(29) $$\sum_{r=1}^{\infty} J(P_{\nu_{r+1}+1}\mathfrak{c}, M_r, N_r) < \infty$$

hold.

First let $r = 1$. By virtue of (26) there exists a natural number N_1 for which

$$J(\mathfrak{c}, 1, N_1) \geqq 1.$$

Applying Lemma 3, there exists another natural number ν_2 such that

$$J(P_{\nu_2+1}\mathfrak{c}, 1, N_1) \leqq \frac{1}{2}.$$

Now $r \geqq 1$ being arbitrary, we assume that $M_\varrho, N_\varrho, \nu_{\varrho+1}$ with $\varrho = 1, 2, \cdots, r-1$ are already defined. According to Lemma 4 there exists a natural number $M_r > N_{r-1}$ such that for every $N > M_r$ we have

(30) $$J(P^{\nu_r}\mathfrak{c}, M_r, N) \leqq \frac{1}{2^r}.$$

By (26) we can choose a natural number $N_r > M_r$ such that

$$J(\mathfrak{c}, M_r, N_r) \geqq 1.$$

(30) holds if N is replaced by N_r in it. Finally using Lemma 3, we obtain a natural number ν_{r+1} for which

$$J(P_{\nu_{r+1}+1}\mathfrak{c}, M_r, N_r) \leqq \frac{1}{2^r}.$$

Thus M_r, N_r and ν_{r+1} will be defined by induction for every $r \geqq 1$ in such a manner that the relations (27), (28) and (29) will be satisfied.

After these preliminaries, we begin with applying Lemma 11 by choosing subsequently M_r and N_r $(r=1, 2, \cdots)$ (instead of M and N). Denote by $\{\psi_n^{(r)}(x)\}_1^\infty$ the corresponding ONS of step functions in $[0, 1]$ and by E_r $(r=1, 2, \cdots)$ the corresponding intervals in the sense of Lemma 11. That is, for every $r \geqq 1$ we have the following properties:

$$\max_{M_r \leqq i < j \leqq N_r} |t_j^{(r)}(x) - t_i^{(r)}(x)| \geqq 2 \tag{31}$$

in the points of the interval $E_r \subseteqq [0, \frac{1}{2}]$ with

$$|E_r| \geqq K_1 \min\left(\frac{1}{2}, J(\mathfrak{c}, M_r, N_r)\right) = \frac{K_1}{2}, \tag{32}$$

where $t_i^{(r)}(x)$ denotes the ith T-mean of the series $\Sigma c_n \psi_n^{(r)}(x)$.

We are going to define a system $\{\Phi_n(x)\}_1^\infty$ of orthonormal step functions in $[0, 1]$, and a stochastically independent sequence $\{F_r\}_1^\infty$ of simple sets [6]) having the following properties: for every $x \in F_r$ there exists a point $y \in E_r$ for which

$$\max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=\nu_r+1}^{\nu_{r+1}} (A_{jn} - A_{in}) c_n \Phi_n(x) \right| = \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=\nu_r+1}^{\nu_{r+1}} (A_{jn} - A_{in}) c_n \psi_n^{(r)}(y) \right|, \tag{33}$$

and

$$|F_r| = |E_r| \qquad (r = 1, 2, \ldots). \tag{34}$$

The construction will be accomplished by recurrence with respect to r. First, let $r=1$. Writing

$$\Phi_n(x) = \psi_n^{(1)}(x) \qquad (n = 1, 2, \ldots, \nu_2),$$

and

$$F_1 = E_1,$$

we can see that (33) and (34) are satisfied.

Now we suppose that all the orthonormal step functions $\Phi_n(x)$ with $n=1, 2, \cdots, \nu_r$ and the stochastically independent simple sets F_ϱ with $\varrho=1, 2, \cdots, r-1$ are already determined and satisfy (33) and (34). Then we can divide $[0, 1]$ into a finite number of subintervals $I_1, I_2, \cdots, I_Q$, in which every function $\Phi_n(x)$ $(n \leqq \nu_r)$ remains constant and every simple set F_ϱ $(\varrho \leqq r-1)$ is the union

[6]) A set F is called simple if it is the union of finitely many non-overlapping intervals.

of a finite number of I_q $(1 \leqq q \leqq Q)$. Let I'_q, I''_q denote the two halves of the interval I_q. Now let us put for $v_r < n \leqq v_{r+1}$

$$\Phi_n(x) = \sum_{q=1}^{Q} \psi_n^{(r)}(I'_q; x) - \sum_{q=1}^{Q} \psi_n^{(r)}(I''_q; x),$$

and

$$F_r = \bigcup_{q=1}^{Q} (E_r(I'_q) \cup E_r(I''_q)),$$

where $f(I; x)$ denotes the function arising from $f(x)$ as the result of the linear transformation of the interval $[0, 1]$ into its subinterval $I=[u, v]$, i.e.

$$f(I; x) = \begin{cases} f\left(\dfrac{x-u}{v-u}\right) & \text{if} \quad x \in (u, v), \\ 0 & \text{otherwise}; \end{cases}$$

furthermore, let $E(I)$ denote the image set of E arising from this transformation. It is obvious that the step functions $\Phi_n(x)$ with $n=1, 2, \cdots, v_{r+1}$ are orthonormal, the simple sets F_ϱ with $\varrho=1, 2, \cdots, r$ are stochastically independent, (33) holds for r, and

$$|F_r| = \sum_{q=1}^{Q} (|E_r(I'_q)| + |E_r(I''_q)|) = |E_r| \sum_{q=1}^{Q} (|I'_q| + |I''_q|) = |E_r|,$$

i.e. (34) is also satisfied for r. Thus $\{\Phi_n(x)\}_1^\infty$ and $\{F_r\}_1^\infty$ will be given by induction.

To finish the proof of the necessity, we have to show that the series

$$\sum_{n=1}^{\infty} c_n \Phi_n(x) \tag{35}$$

fails at almost every point x to be T-summable. For the sake of simplicity, let us denote the ith T-mean of (35) by $T_i(x)$. Taking into account (33), let us consider the following inequality for every $r \geqq 1$

$$\max_{M_r \leqq i < j \leqq N_r} |T_j(x) - T_i(x)| \geqq \max_{M_r \leqq i < j \leqq N_r} |t_j^{(r)}(y) - t_i^{(r)}(y)| -$$

$$- \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=1}^{v_r} (A_{jn} - A_{in}) c_n \Phi_n(x) \right| - \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=v_{r+1}+1}^{\infty} (A_{jn} - A_{in}) c_n \Phi_n(x) \right| -$$

$$- \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=1}^{v_r} (A_{jn} - A_{in}) c_n \psi_n^{(r)}(y) \right| - \max_{M_r \leqq i < j \leqq N_r} \left| \sum_{n=v_{r+1}+1}^{\infty} (A_{jn} - A_{in}) c_n \psi_n^{(r)}(y) \right|,$$

where $x \in F_r$ and $y \in E_r$ in the sense of (33). We show that the last four maxima on the right-hand side of this inequality tend to 0 as $r \to \infty$. In fact, this follows by virtue of (28) and (29), using B. LEVI's theorem. More precisely, there exists a set

G of measure zero such that for every $x \in [0, 1] - G$ we have

$$\overline{\lim_{r\to\infty}} \left(\max_{M_r \leqq i < j \leqq N_r} |T_j(x) - T_i(x)| \right) \geqq \overline{\lim_{r\to\infty}} \left(\max_{M_r \leqq i < j \leqq N_r} |t_j^{(r)}(y) - t_i^{(r)}(y)| \right).$$

Since the sets F_r are stochastically independent, by (32) and (34), we get

$$\left| \overline{\lim_{r\to\infty}} F_r \right| = 1.$$

Thus, on account of (31), we obtain that

$$\overline{\lim_{r\to\infty}} \left(\max_{M_r \leqq i < j \leqq N_r} |T_j(x) - T_i(x)| \right) \geqq 2$$

holds whenever

$$x \in \overline{\lim_{r\to\infty}} F_r - G,$$

that is, for almost every $x \in [0, 1]$.

The proof of the necessity is now complete.

To accomplish the proof of Theorem 1, we have to show that the assertions concerning $\|\mathfrak{c}\|$ are also fulfilled. Let us define the usual vector operations in B as follows:

$$\mathfrak{c} + \mathfrak{d} = \{c_n + d_n\}_1^\infty, \quad \alpha\mathfrak{c} = \{\alpha c_n\}_1^\infty.$$

It is obvious that B is a linear space. From Lemma 1 we infer

$$\left\{ \sum_{n=1}^{\infty} c_n^2 \right\}^{1/2} \leqq \|\mathfrak{c}\| \leqq K \sum_{n=1}^{\infty} |c_n|. \tag{36}$$

$\|\mathfrak{c}\|$ is a norm in B, for (i) $\|\mathfrak{c}\| = 0$ if and only if $c_n = 0$ for every n; (ii) $\|\alpha\mathfrak{c}\| = |\alpha| \|\mathfrak{c}\|$ for every real number α; (iii) $\|\mathfrak{c} + \mathfrak{d}\| \leqq \|\mathfrak{c}\| + \|\mathfrak{d}\|$. (i) follows from (36), (ii) is obvious, (iii) follows from Lemma 5.

We prove that B is a complete space. For this purpose, let $\mathfrak{c}_m = \{c_{mn}\}_{n=1}^\infty \in B$ $(m = 1, 2, \cdots)$ be for which

$$\|\mathfrak{c}_{m'} - \mathfrak{c}_{m''}\| \to 0 \qquad (m', m'' \to \infty).$$

By virtue of (36), we get for every n

$$c_{mn} \to c_n \qquad (m \to \infty).$$

Let ε be an arbitrary positive real number. According to the definition of the norm, we have

$$I(\mathfrak{c}_{m'} - \mathfrak{c}_{m''}, N) \leqq \varepsilon^2 \qquad (m', m'' \geqq \mu(\varepsilon))$$

for every N, and, by virtue of Lemma 2, for every ν

$$I\left(P^\nu(\mathfrak{c}_{m'} - \mathfrak{c}_{m''}), N\right) \leqq \varepsilon^2 \qquad (m', m'' \geqq \mu(\varepsilon)).$$

For m' fixed and m'' tending to infinity, by Lemma 7, we get

$$I\left(P^\nu(\mathfrak{c}_{m'} - \mathfrak{c}), N\right) \leqq \varepsilon^2 \qquad (m' \geqq \mu(\varepsilon))$$

for every ν and N. Hence, applying Lemma 3, we obtain

$$I(\mathfrak{c}_{m'}-\mathfrak{c}, N) \leqq \varepsilon^2 \qquad (m' \geqq \mu(\varepsilon))$$

for every N, where $\mathfrak{c}=\{c_n\}_1^\infty$, and consequently

$$\|\mathfrak{c}_{m'}-\mathfrak{c}\| \leqq \varepsilon \qquad (m' \geqq \mu(\varepsilon)).$$

So we have, by (iii), $\mathfrak{c}\in B$ and, moreover,

$$\|\mathfrak{c}_m-\mathfrak{c}\| \to 0 \qquad (m\to\infty),$$

which was to be proved.

This concludes the proof of Theorem 1.

Proof of Theorem 2. If (1) is T-summable a.e. for every ONS in [0, 1], then by virtue of Theorem 1, we have $\|\mathfrak{c}\|<\infty$. Let us consider an arbitrary ONS $\{\varphi_n(x)\}$ by [0, 1], and denote by $t_i(x)$ the ith T-mean of (1). From Lemma 10, applying *B.* Levi's theorem, we get that the series

$$\sum_{r=1}^{\infty}\left(t_{N_r}(x)-f(x)\right)^2$$

converges a.e. Let us denote by $F(x)$ the positive square root of the sum of this er It is obvious that $F(x)$ is a square integrable function, the square integral of which depends only on the coefficients c_n. By (14), it follows that the function

$$G(x)=\left\{\sum_{r=1}^{\infty}\left(\max_{N_{r-1}\leqq i<j\leqq N_r}|t_j(x)-t_i(x)|\right)^2\right\}^{1/2}$$

is square integrable; its square integral depends only on the coefficients c_n. Let be an arbitrary index with $N_{r-1}<i\leqq N_r$. It is clear that

$$|t_i(x)|\leqq|t_i(x)-t_{N_r}(x)|+|t_{N_r}(x)-f(x)|+|f(x)|\leqq G(x)+F(x)+|f(x)|.$$

This completes the proof.

§3. Proofs of Theorems 3—5

Theorem 3 follows immediately from Lemma 9.

Proof of Theorem 4. Let ε be a positive real number, given in advance, furthermore, let $\{\varphi_n(x)\}$ be an arbitrary ONS in [0, 1]. We denote by $s_k^{(m)}(x)$ the kth partial sum of the series

$$\sum_{n=1}^{\infty}c_{mn}\varphi_n(x)$$

and by $t_i^{(m)}(x)$ the ith T-mean. By Theorem 3, $\|\mathfrak{c}_1\|<\infty$ implies $\|\mathfrak{c}_m\|<\infty$ and so

$\mathfrak{c}_m \in l^2$ for every m. By the Riesz—Fischer theorem there exists a square integrable function $f_m(x)$ such that $\{s_k^{(m)}(x)\}$ converges in the mean to $f_m(x)$ as $k \to \infty$, and so does $\{t_i^{(m)}(x)\}$ as $i \to \infty$.

Since $\|\mathfrak{c}_1\| < \infty$, by virtue of Lemma 10 there exists an increasing sequence $\{N_r\}$ of integers such that $N_0 = 1$,

$$\sum_{r=1}^{\infty} \sum_{n=1}^{\infty} (A_{N_r,n} - 1)^2 c_{1n}^2 < \infty, \tag{37}$$

and

$$\sum_{r=1}^{\infty} J(\mathfrak{c}_1, N_{r-1}, N_r) < \infty. \tag{38}$$

Let us consider the following inequalities:

$$\max_{1 \leqq i \leqq N} |t_i^{(m)}(x)| \leqq |f_m(x)| + \left\{ \sum_{r=1}^{\infty} \left(f_m(x) - t_{N_r}^{(m)}(x) \right)^2 \right\}^{1/2} +$$

$$+ \left\{ \sum_{r=1}^{\infty} \left(\max_{N_{r-1} \leqq i < j \leqq N_r} |t_j^{(m)}(x) - t_i^{(m)}(x)| \right)^2 \right\}^{1/2},$$

whence

$$\int_0^1 \left(\max_{1 \leqq i \leqq N} |t_i^{(m)}(x)| \right)^2 dx \leqq 3 \left(\sum_{n=1}^{\infty} c_{mn}^2 + \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} (A_{N_r,n} - 1)^2 c_{mn}^2 + \sum_{r=1}^{\infty} J(\mathfrak{c}_m, N_{r-1}, N_r) \right)$$

for every $m \geqq 1$ and $N \geqq 1$. By (37) and (38), we can choose the natural numbers ϱ_0 and ν_0 so that

$$\sum_{n=\varrho_0+1}^{\infty} c_{1n}^2 \leqq \varepsilon^2, \qquad \sum_{r=\varrho_0+1}^{\infty} \sum_{n=1}^{\infty} (A_{N_r,n} - 1)^2 c_{1n}^2 \leqq \varepsilon^2,$$

$$\sum_{r=\varrho_0+1}^{\infty} J(\mathfrak{c}_1, N_{r-1}, N_r) \leqq \varepsilon^2, \qquad \sum_{r=1}^{\varrho_0} \sum_{n=\nu_0+1}^{\infty} (A_{N_r,n} - 1)^2 c_{1n}^2 \leqq \varepsilon^2$$

are satisfied. The coefficients c_{mn} being decreasing in m for every fixed n, we obtain

$$\int_0^1 \left(\max_{1 \leqq i \leqq N} |t_i^{(m)}(x)| \right)^2 dx \leqq$$

$$\leqq 3 \left(\sum_{n=1}^{\varrho_0} c_{mn}^2 + \sum_{r=1}^{\varrho_0} \sum_{n=1}^{\nu_0} (A_{N_r,n} - 1)^2 c_{mn}^2 + \sum_{r=1}^{\varrho_0} J(\mathfrak{c}_m, N_{r-1}, N_r) \right) + 12\varepsilon^2. \tag{39}$$

By a simple calculation we get

$$\sum_{r=1}^{\varrho_0} J(\mathfrak{c}_m, N_{r-1}, N_r) \leqq 2 \sum_{r=1}^{\varrho_0} J(P^\lambda \mathfrak{c}_m, N_{r-1}, N_r) + 2 \sum_{r=1}^{\varrho_0} J(P_{\lambda+1} \mathfrak{c}_m, N_{r-1}, N_r), \tag{40}$$

where the natural number λ is fixed in such a manner that

$$(41)\quad \sum_{r=1}^{\varrho_0} J(P_{\lambda+1}\mathfrak{c}_m, N_{r-1}, N_r) \leqq 4 \sum_{r=1}^{\varrho_0} I(P_{\lambda+1}\mathfrak{c}_m, N_r) \leqq 4\varrho_0 I(P_{\lambda+1}\mathfrak{c}_1, N_{\varrho_0}) \leqq \varepsilon^2.$$

Here we took Lemma 9 and Lemma 3 into consideration.

By (39), (40) and (41), on account of Lemma 7, we get that there exists a natural number $\mu(\varepsilon)$ such that

$$\int_0^1 \left(\max_{1\leqq i\leqq N} |t_i^{(m)}(x)|\right)^2 dx \leqq 16\varepsilon^2 \qquad (m \geqq \mu(\varepsilon)).$$

Since $\{\varphi_n(x)\}$ is an arbitrary ONS, thus we obtain for every N

$$I(\mathfrak{c}_m, N) \leqq 16\varepsilon^2 \qquad (m \geqq \mu(\varepsilon)),$$

and consequently

$$\|\mathfrak{c}_m\| \leqq 4\varepsilon \qquad (m \geqq \mu(\varepsilon)),$$

which is what had to be proved.

Proof of Theorem 5. If $\mathfrak{c}\in B$ then, according to Theorem 4, we have

$$\|P^\nu \mathfrak{c} - \mathfrak{c}\| \to 0 \qquad (\nu\to\infty).$$

Hence the class of all the finite sequences is everywhere dense in B. Applying the continuity we infer that every finite sequence can be approximated, as closely as we wish, by a finite sequence of rational numbers. But all the finite sequences of rational numbers form a countable set. So we have proved that B is separable.

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A note on the strong T-summation of orthogonal series

By FERENC MÓRICZ in Szeged

1. Let $\{\varphi_k(x)\}$ $(k=0, 1, \cdots)$ be an orthonormal system on the finite interval (a, b). We shall consider series

$$\sum_{k=0}^{\infty} a_k \varphi_k(x) \tag{1}$$

with real coefficients satisfying

$$\sum_{k=0}^{\infty} a_k^2 < \infty. \tag{2}$$

By the Riesz—Fischer theorem, the series (1) converges in the mean to a square-integrable function $f(x)$. We denote the kth partial sum of the series (1) by $s_k(x)$.

Let $T=(\alpha_{ik})$ $(i, k=0, 1, \cdots)$ be a double infinite matrix of real numbers. We say that the series (1) is *T-summable* to $f(x)$ at the point $x \in (a, b)$ if

$$t_i = \sum_{k=0}^{\infty} \alpha_{ik} s_k(x)$$

exists for all i (except perhaps finitely many of them), and

$$\lim_{i\to\infty} t_i(x) = f(x).$$

The series (1) is called *strongly T-summable* at the point x if the relation

$$\lim_{i\to\infty} \sum_{k=0}^{\infty} \alpha_{ik} (s_k(x) - f(x))^2 = 0$$

holds.

A T-summation process is called *permanent* if $\lim_{k\to\infty} s_k = s$ always implies $\lim_{i\to\infty} t_i = s$. Necessary and sufficient conditions for the permanence of a summation process are well known. (See ALEXITS [1], p. 65.)

2. In the most frequently used cases T-summability and strong T-summability of the series (1) coincide under the condition (2), up to sets of measure zero. For the classical $(C, 1)$-summation process this was proved by ZYGMUND [9] (see also

TANDORI [8]), for $(C, \beta>0)$-summation by SUNOUCHI [7], and for Riesz summation by MEDER [4] and LEINDLER [2]. (In the latter case

$$\alpha_{ik} = \frac{\lambda_{k+1}-\lambda_k}{\lambda_{i+1}} \quad \text{for} \quad k \leqq i, \quad \alpha_{ik}=0 \quad \text{for} \quad k>i,$$

where $\{\lambda_i\}$ is a strictly increasing sequence of positive real numbers with $\lambda_0=0$ and $\lambda_i \to \infty$.) Finally, for the de la Vallée Poussin summation, this was proved also by LEINDLER [3]. (In this case

$$\alpha_{ik} = \frac{1}{\mu_i} \quad \text{if} \quad k = i-\mu_i+1, \quad i-\mu_i+2, \ldots, i;$$

$$\alpha_{ik} = 0 \quad \text{if} \quad k = 0, 1, \ldots, i-\mu_i; \quad i+1, \quad i+2, \ldots,$$

where $\{\mu_i\}$ is an increasing sequence of natural numbers with $\mu_{i+1}-\mu_i \leqq 1$.)

3. These particular results raise the following question: does, under condition (2), T-summability of the series (1) almost everywhere imply strong T-summability for any T-process?

In this paper we show that the answer is in general negative. We prove the following

Theorem. *There exist a uniformly bounded orthonormal system $\{\Phi_k(x)\}$ in $(0, 1)$, a sequence $\{c_k\}$ of coefficients and a permanent T-summation process such that*

$$\sum_{k=0}^{\infty} c_k^2 < \infty$$

is satisfied, the orthonormal series

$$\sum_{k=0}^{\infty} c_k \Phi_k(x) \tag{3}$$

is T-summable almost everywhere, but the relation

$$\varlimsup_{i\to\infty} \sum_{k=0}^{\infty} \alpha_{ik} |s_k(x)-f(x)|^\gamma = \infty \tag{4}$$

holds almost everywhere in $(0, 1)$ *for any* $\gamma>0$.

The proof will be accomplished by direct construction, the T-summation in question being defined by a method due to MENCHOFF [6].

4. We require some lemmas. In the sequel, we use $C, C_1, C_2, \cdots$ to denote positive constants.

Lemma 1. (MENCHOFF [5]) *Let* $\nu > 3$ *be a natural number and let* $C > 1$. *Then there exists in* $(-1, C)$ *a system* $\{\psi_{k\nu}(x)\}$ $(1 \leqq k \leqq \nu^2)$ *of orthonormal step functions with the following properties:*

(i) $|\psi_{k\nu}(x)| \leqq C_1$ $(1 \leqq k \leqq \nu^2,\ -1 \leqq x \leqq C)$;

(ii) *for every point* $x \in (\frac{1}{2}, 1)$ *there exists an index* $l(x)$ *depending on* x $\left(1 \leqq l(x) \leqq \nu^2\right)$ *such that*

$$\sum_{k=1}^{l(x)} \psi_{k\nu}(x) \geqq C_2 \nu \log \nu.$$

Let us define another system $\{\chi_{k\nu}(x)\}$ $(1 \leqq k \leqq 2\nu^2)$ of orthonormal step functions in $(-2, C)$ as follows:

$$\chi_{k\nu}(x) = \chi_{\nu^2+k,\nu}(x) = \frac{1}{\sqrt{2}} \psi_{k\nu}(x) \qquad (1 \leqq k \leqq \nu^2,\ -1 \leqq x \leqq C), \tag{5}$$

$$\chi_{k\nu}(x) = \frac{1}{\sqrt{2}} r_k(x+2), \quad \chi_{\nu^2+k,\nu}(x) = -\frac{1}{\sqrt{2}} r_k(x+2) \qquad (1 \leqq k \leqq \nu^2,\ -2 \leqq x < -1),$$

where $r_k(x) = \operatorname{sign} \sin 2^k \pi x$ denotes the kth Rademacher function $(k = 0, 1, \cdots)$. By virtue of Lemma 1 it is clear that

$$|\chi_{k\nu}(x)| \leqq C_3 \qquad (1 \leqq k \leqq 2\nu^2,\ -2 \leqq x \leqq C);$$

furthermore, for every point $x \in (\frac{1}{2}, 1)$ there exists an index $l(x)$ $\left(1 \leqq l(x) \leqq \nu^2\right)$ such that

$$\sum_{k=1}^{l(x)} \chi_{k\nu}(x) = \sum_{k=\nu^2+1}^{\nu^2+l(x)} \chi_{k\nu}(x) \geqq C_4 \nu \log \nu. \tag{6}$$

This construction can also be found in the cited paper of MENCHOFF [6].

5. Proof of the theorem. Let $g(y)$ be an arbitrary function defined in $(-2, C)$ and let $I = (u, v)$ be an arbitrary finite interval. We proceed from the interval I to the interval $(-2, C)$ by means of the linear transformation

$$y = -2 + \frac{x-u}{v-u}(2+C) \quad (u \leqq x \leqq v,\ -2 \leqq y \leqq C),$$

and put

$$g(I; x) = \begin{cases} \sqrt{2+C}\, g(y) & \text{if } \ u \leqq x \leqq v, \\ 0 & \text{elsewhere.} \end{cases}$$

Further, let $E(I)$ denote the image set of an arbitrary set $E \subset (-2, C)$ arising from this transformation. It is obvious that

$$\int_u^v g^2(I; x)\, dx = |I| \int_{-2}^{C} g^2(y)\, dy.\ ^{1)}$$

---

[1]) $|I|$ denotes the Lebesgue measure of the set I.

We are going to construct the system $\{\Phi_k(x)\}$ and an auxiliary system $\{\Psi_k(x)\}$ which has an important role in the proof. Let $\{\nu_r\}$ be any sequence of natural numbers, with $\nu_r > 3$ $(r=1, 2, \cdots)$, and let

$$N_0 = 0, \quad N_r = 2\sum_{\varrho=1}^{r} \nu_\varrho^2, \quad N_r' = N_{r-1} + \nu_r^2 \qquad (r=1, 2, \ldots).$$

First we set

$$\Phi_k(x) = \Psi_k(x) = r_k(x) \qquad (k = 0, 1, \ldots, N_1; 0 \leqq x \leqq 1).$$

Now $r > 1$ being arbitrary, we assume that the step functions $\Phi_k(x)$, $\Psi_k(x)$ $(k=0, 1, \cdots, N_{r-1})$ are already defined. Then we divide $(0, 1)$ into a finite number of mutually disjoint subintervals $I_1, I_2, \cdots, I_s$, in which every function $\Phi_k(x)$, $\Psi_k(x)$ with $k \leqq N_{r-1}$ is constant. Let I_σ', I_σ'' denote the two halves of the interval I_σ, and set

$$\Phi_k(x) = \begin{cases} \chi_{k-N_{r-1}, \nu_r}(I_\sigma'; x) & \text{if} \quad x \in I_\sigma' \\ -\chi_{k-N_{r-1}, \nu_r}(I_\sigma''; x) & \text{if} \quad x \in I_\sigma'' \end{cases} \quad (\sigma = 1, 2, \ldots, s;\ N_{r-1} < k \leqq N_r);$$

$$\Psi_k(x) = \begin{cases} \dfrac{1}{\sqrt{2+C}}\, r_{k-N_{r-1}}(\tilde I_\sigma; x) & \text{if} \quad x \in \tilde I_\sigma \quad \text{and} \quad N_{r-1} < k \leqq N_r', \\ -\dfrac{1}{\sqrt{2+C}}\, r_{k-N_r'}(\tilde I_\sigma; x) & \text{if} \quad x \in \tilde I_\sigma \quad \text{and} \quad N_r' < k \leqq N_r, \end{cases}$$

where $\tilde I_\sigma$ can be either I_σ' or I_σ'' $(\sigma = 1, 2, \cdots, s)$. It is clear that these functions are also step functions.

Set $E_1 = (-2, -1)$, $E_2 = (-1, C)$ and $E_3 = (\frac{1}{2}, 1)$; furthermore, write

$$G_r'(1) = \bigcup_{\sigma=1}^{s} E_1(I_\sigma'), \quad G_r''(1) = \bigcup_{\sigma=1}^{s} E_1(I_\sigma''),$$

and

$$G_r(i) = \bigcup_{\sigma=1}^{s} (E_i(I_\sigma') \cup E_i(I_\sigma'')) \qquad (i=2, 3).$$

It is obvious that the interval $(0, 1)$ is the union of the mutually disjoint subsets $G_r'(1)$, $G_r''(1)$ and $G_r(2)$, and that

$$(7) \qquad |G_r(3)| = \frac{1}{2(2+C)} \qquad (r=1, 2, \ldots).$$

We can easily prove that the system $\{\Phi_k(x)\}$ as constructed from the previously defined functions is orthonormal and uniformly bounded. Furthermore, the system $\{\Psi_k(x)\}$ can be divided into two subsystems, both of which are orthonormal. More exactly, MENCHOFF [6] proved the following

Lemma 2. *Let* $\{\Psi_k(x)\}$ *be the system of functions in* $(0, 1)$ *defined above, and set*

$$S' = \bigcup_{r=1}^{\infty} \{\Psi_k(x) : N_{r-1} < k \leqq N'_r\}, \qquad S'' = \bigcup_{r=1}^{\infty} \{\Psi_k(x) : N'_r < k \leqq N_r\}.$$

Then both S' and S'' are orthonormal convergence systems[2]).

6. We define the matrix $T = (\alpha_{ik})$ $(i, k = 0, 1, \cdots)$ as follows

$$\alpha_{00} = 1 \quad \text{and} \quad \alpha_{0k} = 0 \quad \text{for} \quad k \geqq 1,$$

and in general, for an arbitrary natural number $r(\geqq 1)$, we distinguish two subcases: if $N_{r-1} < i \leqq N'_r$, then we set

$$\alpha_{ii} = \alpha_{i,\, v_r^2 + i} = \frac{1}{2} \quad \text{and} \quad \alpha_{ik} = 0 \quad \text{otherwise};$$

if $N'_r < i \leqq N_r$, then

$$\alpha_{i, N_r} = 1 \quad \text{and} \quad \alpha_{ik} = 0 \quad \text{otherwise}.$$

From the definition of the matrix T we can immediately infer the permanence of the T-summation process.

7. We define the sequence $\{c_k\}$ $(c_0 = 0)$ of coefficients as follows

$$c_k = \begin{cases} p_r & \text{if} \quad N_{r-1} < k \leqq N'_r, \\ -p_r & \text{if} \quad N'_r < k \leqq N_r \end{cases} \qquad (r = 1, 2, \ldots),$$

where the sequence $\{v_r\}$ of natural numbers and the sequence $\{p_r\}$ of positive real numbers are chosen such that the relations

(8) $$\sum_{r=1}^{\infty} p_r v_r < \infty,$$

and

(9) $$\lim_{r \to \infty} p_r v_r \log v_r = \infty$$

are satisfied. An appropriate choice is for example

$$v_r = 2^{r^3} \quad \text{and} \quad p_r = \frac{1}{r^2 v_r} \qquad (r = 1, 2, \ldots).$$

---

[2]) An orthonormal system $\{\varphi_k(x)\}$ is called a convergence system if every series $\Sigma a_k \varphi_k(x)$ whose coefficients satisfy the condition (2) is convergent almost everywhere.

8. By (8) we can easily see that

$$\sum_{k=0}^{\infty} c_k^2 = \sum_{r=1}^{\infty} \sum_{k=N_{r-1}+1}^{N_r} c_k^2 = 2 \sum_{r=1}^{\infty} p_r^2 v_r^2 \leqq C_5 \sum_{r=1}^{\infty} p_r v_r < \infty.$$

We show that (8) implies also the convergence of the partial sums $\{s_{N_r}(x)\}$ and $\{s_{N_r'}(x)\}$ of the series (3) almost everywhere. On account of

$$\sum_{r=1}^{\infty} \int_0^1 |s_{N_r}(x) - s_{N_{r-1}}(x)| dx \leqq \sum_{r=1}^{\infty} \left\{ \int_0^1 (s_{N_r}(x) - s_{N_{r-1}}(x))^2 dx \right\}^{\frac{1}{2}} =$$

$$= \sum_{r=1}^{\infty} \left\{ \sum_{k=N_{r-1}+1}^{N_r} c_k^2 \right\}^{\frac{1}{2}} = \sqrt{2} \sum_{r=1}^{\infty} p_r v_r < \infty,$$

we infer, by applying the theorem of B. Levi, that the sequence $\{s_{N_r}(x)\}$ is convergent. The convergence of $\{s_{N_r'}(x)\}$ almost everywhere follows in the same way.

9. Now we are able to prove the T-summability of the series (3) almost everywhere. On the one hand, if $N_r' < i \leqq N_r$, then we have

$$t_i(x) = s_{N_r}(x);$$

on the other hand, if $N_{r-1} < i \leqq N_r'$, then

$$t_i(x) = \frac{1}{2} s_i(x) + \frac{1}{2} s_{i+v_r^2}(x) = \frac{1}{2} s_{N_{r-1}}(x) + \frac{1}{2} s_{N_r'}(x) + \left\{ \sum_{k=N_{r-1}+1}^{i} + \sum_{k=N_r'+1}^{i+v_r^2} \right\} c_k \Phi_k(x).$$

For the sake of brevity, we write

$$R(r, i; x) = \left\{ \sum_{k=N_{r-1}+1}^{i} + \sum_{k=N_r'+1}^{i+v_r^2} \right\} c_k \Phi_k(x).$$

For our purpose it is enough to show that $R(r, i; x)$ tends to 0 almost everywhere in (0, 1) as $r \to \infty$. Taking into account the definition of the coefficients c_k and (5), we can see that the $R(r, i; x)$ equals 0 at every point $x \in G_r(2)$. In case $x \in G_r'(1) \cup \cup G_r''(1)$, we get by a simple calculation that

$$\Phi_k(x) = \frac{\sqrt{2+C}}{\sqrt{2}} \Psi_k(x) \quad \text{if} \quad x \in G_r'(1), \qquad \Phi_k(x) = -\frac{\sqrt{2+C}}{\sqrt{2}} \Psi_k(x) \quad \text{if} \quad x \in G_r''(1)$$

$(N_{r-1} < k \leqq N_r', \ r = 1, 2, \cdots)$. Hence we can write

$$R(r, i; x) = \pm \frac{\sqrt{2+C}}{\sqrt{2}} \left\{ \sum_{k=N_{r-1}+1}^{i} + \sum_{k=N_r'+1}^{i+v_r^2} \right\} c_k \Psi_k(x),$$

according as $x \in G_r'(1)$ or $x \in G_r''(1)$. Applying Lemma 2, we infer that $R(r, i; x)$ tends to 0 almost everywhere as $r \to \infty$.

10. To accomplish the proof, we have to show that (4) is also satisfied. Let us consider the sets $G_r(3)$ $(r=1, 2, \cdots)$. According to the definition of the intervals I'_σ, I''_σ $(\sigma=1, 2, \cdots, s)$ and $G_r(3)$, we can easily see that the sets $G_r(3)$ are stochastically independent. Applying the Borel—Cantelli lemma we get, by virtue of (7),

$$\left|\overline{\lim_{r\to\infty}}\, G_r(3)\right| = 1. \tag{10}$$

Let $N_{r-1}<i\leqq N_r$. By looking at the inequality

$$|a-b|^\gamma \geqq C(\gamma)|a|^\gamma - |b|^\gamma \qquad (\gamma>0),\ ^{3)}$$

where $C(\gamma)$ denotes a positive constant depending only on γ, we obtain the estimate

$$\sum_{k=0}^{\infty} \alpha_{ik}|s_k(x)-f(x)|^\gamma = \frac{1}{2}|s_i(x)-f(x)|^\gamma + \frac{1}{2}|s_{i+\nu_r^2}(x)-f(x)|^\gamma \geqq$$

$$\geqq C(\gamma)\left\{\left|\sum_{k=N_{r-1}+1}^{i} c_k\Phi_k(x)\right|^\gamma + \left|\sum_{k=N'_r+1}^{i+\nu_r^2} c_k\Phi_k(x)\right|^\gamma\right\} -$$

$$-\frac{1}{2}|s_{N_{r-1}}(x)|^\gamma - \frac{1}{2}|s_{N'_r}(x)|^\gamma - |f(x)|^\gamma.$$

By virtue of (6), there exists an index $i=l(x)$ $(N_{r-1}<l(x)\leqq N'_r)$ for almost every point $x\in G_r(3)$ such that

$$\sum_{k=0}^{\infty} \alpha_{l(x),k}|s_k(x)-f(x)|^\gamma \geqq C_4 C(\gamma) p_r \nu_r \log \nu_r - C(x)$$

holds, where $C(x)$ is a positive constant depending only on x. Here we again took into consideration that the sequences $\{s_{N_r}(x)\}$ and $\{s_{N'_r}(x)\}$ converge almost everywhere. By (10) this estimate holds at almost every point $x\in(0, 1)$ for infinitely many values of r. Using (9), we get that the relation (4) is satisfied almost everywhere.

We have thus completed the proof of our theorem.

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[3]) If $0<\gamma\leqq 1$ then this inequality follows from $|a+b|^\gamma \leqq |a|^\gamma+|b|^\gamma$, and if $\gamma>1$ then it follows from $|a+b|^\gamma \leqq 2^{\gamma-1}(|a|^\gamma+|b|^\gamma)$.

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Über die starke Summation von Walsh—Fourierreihen

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Einleitung

Wir bezeichnen mit $x=0, x_0 x_1 \cdots x_n \cdots$ die dyadische Entwicklung der Zahl $x \in [0, 1)$, wobei wir festsetzen, daß für $x=p2^{-q}$ alle x_i $(i \geqq q)$ gleich 0 sind.

Das Rademachersche System $\{r_n(x)\}$ ist folgenderweise definiert:

$$r_n(x)=(-1)^{x_n} \qquad (x \in [0, 1);\ n=0, 1, 2, \cdots). \tag{1}$$

Wir bezeichnen mit $\{\psi_n(x)\}$ $(n=0, 1, 2, \cdots)$ das Walshsche Orthogonalsystem, d.h. es ist $\psi_0(x) \equiv 1$, und

$$\psi_n(x) = \prod_{n_i=1} r_i(x) \quad \text{für} \quad n = \sum_{i=0}^{\infty} n_i 2^i \quad (n_i = 0, 1). \tag{2}$$

Auf Grund von (1) und (2) folgt

$$\psi_n(x) = (-1)^{\sum_{i=0}^{\infty} n_i x_i}. \tag{3}$$

N. J. FINE [1] hatte die folgende Operation eingeführt: es sei

$$x \dot{+} y = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{i+1}} \tag{4}$$

$$\text{für } x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}, \qquad y = \sum_{i=0}^{\infty} \frac{y_i}{2^{i+1}} \ (x_i, y_i=0, 1).$$

Es ist leicht zu verifizieren, daß die Gleichung

$$\psi_n(x \dot{+} y) = \psi_n(x)\psi_n(y) \tag{5}$$

mit Ausnahme einer abzählbaren Teilmenge von $[0, 1)$ überall besteht.

Für $n=1, 2, \ldots$ bezeichne $D_n(x)$ die n-te Walsh—Dirichletsche Kernfunktion:

$$D_n(x) = \sum_{\nu=0}^{n-1} \psi_\nu(x) \qquad (n = 1, 2, \ldots) \tag{6}$$

und für $n=0$ sei $D_0(x)\equiv 0$. Es ist leicht zu zeigen, daß für $n=2^k+n'$ $(0<n'\leqq 2^k)$

$$D_n(x) = D_{2^k}(x)+r_k(x)D_{n'}(x) \tag{7}$$

gilt, woraus sich für $n'=2^k$ die Gleichung

$$D_{2^{k+1}}(x) = \prod_{\nu=0}^{k}(1+r_\nu(x)) = \begin{cases} 2^{k+1} & \left(x\in\left[0, \frac{1}{2^{k+1}}\right)\right) \\ 0 & \left(x\in\left[\frac{1}{2^{k+1}}, 1\right)\right) \end{cases} \tag{8}$$

ergibt.

Wir bezeichnen mit $S_n(f; x)$ die n-te Partialsumme der Walsh—Fourier-Entwicklung von $f(x)$, d.h.

$$S_n(f; x) = \sum_{\nu=0}^{n-1} c_\nu(f)\psi_\nu(x) = \int_0^1 f(x\dotplus u)D_n(u)\,du, \quad c_\nu(f) = \int_0^1 f(u)\psi_\nu(u)\,du. \tag{9}$$

In dieser Arbeit werden wir den folgenden Satz beweisen.

Satz. *Für* $f(x)\in L[0, 1]$ *gilt fast überall*

$$h_n(f, x; 2) = \left\{\frac{1}{n}\sum_{\mu=0}^{n-1}|S_\mu(f; x)-f(x)|^2\right\}^{\frac{1}{2}} \to 0 \qquad (n\to\infty).$$

Ähnlicherweise kann man zeigen, daß $h_n(f, x; 2m)\to 0$ $(n\to\infty)$ *für jede positive, ganze Zahl m fast überall besteht.* Daraus folgt $h_n(f, x; r)\to 0$ $(n\to\infty)$ *fast überall für jedes* $r>0$.

A. Zygmund [2] hat einen analogen Satz für trigonometrische Fourierreihen bewiesen.

§1. Hilfssätze

Hilfssatz I. *Es sei*

$$D_n^*(u; l, k) = \sum_{i=l}^{k-1}\psi_n\left(u\dotplus\frac{1}{2^{i+1}}\right)r_i\left(u\dotplus\frac{1}{2^{i+1}}\right)D_{2^i}\left(u\dotplus\frac{1}{2^{i+1}}\right) \tag{1.1}$$

$(0\leqq l<k,\ \ n=0, 1, 2, \ldots)$. *Dann besteht fast überall:*

$$D_n(u) = \tfrac{1}{2}(D_{2^k}(u)-\psi_n(u))+\tfrac{1}{2}D_n^*(u; 0, k) \qquad (n=0, 1, \ldots). \tag{1.2}$$

Beweis von Hilfssatz I. In der Arbeit [3] haben wir bewiesen, daß für

$$n=\sum_{i=0}^{k-1}n_i2^i \qquad (n_i=0, 1)$$

die Gleichung

$$D_n(u) = \psi_n(u)\sum_{i=0}^{k-1}n_ir_i(u)D_{2^i}(u) \qquad (n=0, 1, 2, \ldots) \tag{1.3}$$

besteht. Aus (3) folgt

$$\tfrac{1}{2}\left(1-\psi_n\left(\frac{1}{2^{i+1}}\right)\right) = \tfrac{1}{2}(1-(-1)^{n_i}) = n_i,$$

woraus sich, auf Grund von (1), (5) und (1. 3) die Gleichung

$$(1.4)\qquad D_n(u) = \tfrac{1}{2}\psi_n(u)\sum_{i=0}^{k-1} r_i(u)D_{2^i}(u)+\tfrac{1}{2}\sum_{i=0}^{k-1}\psi_n\left(u\dotplus\frac{1}{2^{i+1}}\right)r_i\left(u\dotplus\frac{1}{2^{i+1}}\right)D_{2^i}(u)$$

fast überall ergibt. Da nach (8) $D_{2^i}(u)=D_{2^i}\left(u\dotplus\frac{1}{2^{i+1}}\right)$ ist, weiterhin aus (8) und (1.3) folgt

$$\sum_{i=0}^{k-1} r_i(u)D_{2^i}(u) = \psi_{2^k-1}(u)D_{2^k-1}(u) = \psi_{2^k-1}(u)\left(D_{2^k}(u)-\psi_{2^k-1}(u)\right) = D_{2^k}(u)-1,$$

so erhalten wir auf Grund von (8) und (1. 1) aus (1. 4) die zu beweisende Gleichung:

$$D_n(u) = \tfrac{1}{2}\psi_n(u)\left(D_{2^k}(u)-1\right)+\tfrac{1}{2}D_n^*(u;0,k) = \tfrac{1}{2}\left(D_{2^k}(u)-\psi_n(u)\right)+\tfrac{1}{2}D_n^*(u;0,k).$$

Hilfssatz II. *Es sei*

$$K_{2^k}(u,v;l) = 2^{-k}\sum_{n=0}^{2^k-1} D_n^*(u;l,k)D_n^*(v;l,k)\qquad (0\leqq l<k,\quad k=0,1,2,\ldots).$$

Dann besteht

$$(1.5)\quad K_{2^k}(u,v;l) = 2^{-k}\sum_{i,j=l}^{k-1} r_i(u)D_{2^i}(u)r_j(v)D_{2^j}(v)D_{2^k}\left(u\dotplus v\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)$$

fast überall in $E_2=\{(u,v)\colon 0\leqq u<1,\ 0\leqq v<1\}$.

Beweis von Hilfssatz II. Aus (5) und (1. 1) folgt, daß

$$2^kK_{2^k}(u,v;l) = \sum_{n=0}^{2^k-1}\left(\sum_{i=l}^{k-1}\psi_n\left(u\dotplus\frac{1}{2^{i+1}}\right)r_i(u)D_{2^i}(u)\right)\left(\sum_{j=l}^{k-1}\psi_n\left(v\dotplus\frac{1}{2^{j+1}}\right)r_j(v)D_{2^j}(v)\right)=$$

$$=\sum_{n=0}^{2^k-1}\sum_{i,j=l}^{k-1}\psi_n\left(u\dotplus v\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)r_i(u)D_{2^i}(u)r_j(v)D_{2^j}(v)=$$

$$=\sum_{i,j=l}^{k-1} r_i(u)D_{2^i}(u)r_j(v)D_{2^j}(v)\sum_{n=0}^{2^k-1}\psi_n\left(u\dotplus v\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)=$$

$$=\sum_{i,j=l}^{k-1} r_i(u)D_{2^i}(u)r_j(v)D_{2^j}(v)D_{2^k}\left(u\dotplus v\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)$$

fast überall in E_2 besteht, womit die Behauptung (1. 5) bewiesen ist.

Zum Beweis unseres Satzes haben wir ein Hilfssatz von S. IGARI [4], [5] nötig, den wir in der folgenden modifizierten Form anwenden.

Hilfssatz III. *Es sei $f(u)\in L[0,1]$. Dann gibt es für jede Zahl $M>\int_0^1 |f(u)|\,du$ ein System $\{I_\nu\}$ $(\nu=1,2,\cdots)$ von dyadischen Intervallen und zwei Funktionen $f_1(u)$ und $w(u)$ derart, daß die folgenden Relationen gelten:*

a) $I_\nu=\left[\frac{a_\nu}{2^{m_\nu}},\frac{a_\nu+1}{2^{m_\nu}}\right)$ $\quad (a_\nu<2^{m_\nu},\quad a_\nu\in G,\quad \nu=1,2,\ldots)$, [1])

$I_\nu\cap I_{\nu'}=\emptyset \qquad (\nu\neq\nu')$;

b) $f(u)=f_1(u)+w(u), \qquad w(u)=\sum_{\nu=1}^{\infty} w_\nu(u)$;

c) $w_\nu(u)=0 \quad (u\notin I_\nu), \qquad \int_{I_\nu} w_\nu(u)\,du=0, \qquad \int_{I_\nu}|w_\nu(u)|\,du\leqq 4M|I_\nu|$; [2])

d) $|f_1(u)|\leqq 2M$ *(fast überall in* $[0,1]$);

e) $\sum_{\nu=1}^{\infty}|I_\nu|\leqq \frac{1}{M}\int_0^1 |f(u)|\,du.$

Auf Grund des Beweises von Lemma 1 in [4] kann man Hilfssatz III leicht zeigen.

Hilfssatz IV. *Es sei $\{I_\nu\}$ $(\nu=1,2,3,\cdots)$ ein System von disjunkten dyadischen Intervallen mit $I_\nu=\left[\frac{a_\nu}{2^{m_\nu}},\frac{a_\nu+1}{2^{m_\nu}}\right)$ $(a_\nu<2^{m_\nu}, a_\nu\in G)$ und es sei weiterhin*

$$(1.6)\qquad \begin{cases} \varphi_\nu(u)=\begin{cases}|I_\nu| & (u\in I_\nu),\\ 0 & (u\in[0,1)-I_\nu)\end{cases} \quad (\nu=1,2,3,\ldots);\\ \Phi_j(u)=\sum_{|I_\nu|\leqq 2^{-j}}\varphi_\nu(u) \qquad (j=0,1,2,\ldots); \end{cases}$$

$$w^*(u)\geqq 0 \text{ für } u\in\bigcup_{\nu=1}^{\infty} I_\nu, \text{ und } w^*(u)=0 \text{ für } u\in[0,1)-\bigcup_{\nu=1}^{\infty} I_\nu;$$

$$\Psi_i(x)=\Psi_i(w^*;x)=$$

$$=\begin{cases}\sum_{j=m_\nu}^{\infty}\int_0^1 w^*(u)D_{2^j}\left(x\dotplus u\dotplus\frac{1}{2^{i+1}}\right)\Phi_{m_\nu}\left(u\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)du & \text{für } x\dotplus\frac{1}{2^{i+1}}\in I_\nu,\\ 0 \quad \text{für } x\dotplus\frac{1}{2^{i+1}}\notin\bigcup_{\nu=1}^{\infty} I_\nu.\end{cases}$$

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[1]) G bedeutet die Menge der nichtnegativen ganzen Zahlen.

[2]) $|I_\nu|$ bedeutet das Lebesguesche Maß von I_ν.

Dann besteht in fast allen Punkten $x \in F = [0, 1) - \bigcup_{\nu=1}^{\infty} I_\nu$

a) $\displaystyle\sum_{i=0}^{\infty} \int_0^1 D_{2^i}^2(x \dotplus u) \Phi_0(u)\, du = N_1(x) < \infty,$

b) $\displaystyle\sum_{i=0}^{\infty} 2^i \sum_{j=i+1}^{\infty} \int_0^1 D_{2^j}\left(x \dotplus u \dotplus \frac{1}{2^{i+1}}\right) \Phi_j(u)\, du = N_2(x) < \infty,$

c) $\displaystyle\sum_{i=0}^{\infty} 2^i \Psi_i(x) = N_3(x) < \infty.$

Beweis von Hilfssatz IV. Nach dem Satz von Fubini gilt

$$A_1 = \sum_{i=0}^{\infty} \int_F \left(\int_0^1 D_{2^i}^2(x \dotplus u) \Phi_0(u)\, du \right) dx = \sum_{i=0}^{\infty} 2^i \int_0^1 \Phi_0(u) \left(\int_F D_{2^i}(x \dotplus u)\, dx \right) du =$$

$$= \sum_{i=0}^{\infty} 2^i \sum_{\nu=1}^{\infty} \int_{I_\nu} \varphi_\nu(u) \left(\int_F D_{2^i}(x \dotplus u)\, dx \right) du.$$

Da

$$(1.7) \qquad \begin{cases} D_{2^i}(x \dotplus u) = 0 \quad (u \in I_\nu,\ x \notin I_\nu,\ i \geqq m_\nu), \\ \displaystyle\int_F D_{2^i}(x \dotplus u)\, dx \leqq \int_0^1 D_{2^i}(x \dotplus u)\, dx = 1, \end{cases}$$

deshalb ist auf Grund von (1. 6)

$$A_1 = \sum_{\nu=1}^{\infty} \sum_{i=0}^{m_\nu - 1} 2^i \int_{I_\nu} \varphi_\nu(u) \left(\int_F D_{2^i}(x \dotplus u)\, dx \right) du \leqq \sum_{\nu=1}^{\infty} |I_\nu|^2 \sum_{i=0}^{m_\nu - 1} 2^i < \sum_{\nu=1}^{\infty} |I_\nu| < 1,$$

woraus nach dem Satz von B. Levi die Behauptung a) folgt.

Ähnlich ergibt sich b):

$$\sum_{i=0}^{\infty} 2^i \sum_{j=i+1}^{\infty} \int_F \left(\int_0^1 D_{2^j}\left(x \dotplus u \dotplus \frac{1}{2^{i+1}}\right) \Phi_j(u)\, du \right) dx =$$

$$= \sum_{i=0}^{\infty} 2^i \sum_{j=i+1}^{\infty} \int_0^1 \Phi_j(u) \left(\int_F D_{2^j}\left(x \dotplus u \dotplus \frac{1}{2^{i+1}}\right) dx \right) du =$$

$$= \sum_{i=0}^{\infty} 2^i \sum_{j=i+1}^{\infty} \sum_{|I_\nu| \leqq \frac{1}{2^j}} \int_0^1 \varphi_\nu(u) \left(\int_F D_{2^j}\left(x \dotplus u \dotplus \frac{1}{2^{i+1}}\right) dx \right) du =$$

$$= \sum_{\nu=1}^{\infty} \sum_{i=0}^{m_\nu - 1} 2^i \sum_{j=i+1}^{m_\nu} |I_\nu|^2 = \sum_{\nu=1}^{\infty} |I_\nu| \sum_{i=0}^{m_\nu - 1} 2^{i - m_\nu}(m_\nu - i) < \left(\sum_{j=1}^{\infty} j 2^{-j} \right) \sum_{\nu=1}^{\infty} |I_\nu| < 4.$$

Aus (1.6) folgt
$$\int_F \Psi_i(x)\,dx = \int_{F\dot{+}\frac{1}{2^{i+1}}} \Psi_i\left(x\dot{+}\frac{1}{2^{i+1}}\right) dx\,^{3)} =$$
$$=\sum_{\nu=1}^{\infty}\sum_{j\geqq m_\nu}\int_{\left(F\dot{+}\frac{1}{2^{i+1}}\right)\cap I_\nu}\left(\int_0^1 w^*(u)D_{2^j}(x\dot{+}u)\Phi_{m_\nu}\left(u\dot{+}\frac{1}{2^{i+1}}\dot{+}\frac{1}{2^{j+1}}\right)du\right)dx =$$
$$=\sum_{\nu=1}^{\infty}\sum_{j\geqq m_\nu}\int_0^1 w^*(u)\Phi_{m_\nu}\left(u\dot{+}\frac{1}{2^{i+1}}\dot{+}\frac{1}{2^{j+1}}\right)\left(\int_{\left(F\dot{+}\frac{1}{2^{i+1}}\right)\cap I_\nu} D_{2^j}(x\dot{+}u)\,dx\right)du.$$

Da im Falle $x\in I_\nu$, $u\notin I_\nu$, $j\geqq m_\nu$ $D_{2^j}(x\dot{+}u)=0$ gilt, deshalb ist
$$\int_F \Psi_i(x)\,dx=\sum_{\nu=1}^{\infty}\sum_{j\geqq m_\nu}\int_{I_\nu} w^*(u)\Phi_{m_\nu}\left(u\dot{+}\frac{1}{2^{i+1}}\dot{+}\frac{1}{2^{j+1}}\right)\left(\int_{\left(F\dot{+}\frac{1}{2^{i+1}}\right)\cap I_\nu} D_{2^i}(x\dot{+}u)\,dx\right)du =$$
$$=\sum_{\nu=1}^{\infty}\sum_{j\geqq m_\nu}\int_{I_{\nu'}\dot{+}\frac{1}{2^{i+1}}\subset I_\nu}|I_{\nu'}|\int_{I_{\nu'}\dot{+}\frac{1}{2^{i+1}}\dot{+}\frac{1}{2^{j+1}}} w^*(u)\left(\int_{\left(F\dot{+}\frac{1}{2^{i+1}}\right)\cap I_\nu} D_{2^j}(x\dot{+}u)\,dx\right)du.$$

Da endlich
$$\left(F\dot{+}\frac{1}{2^{i+1}}\right)\cap I_\nu=\emptyset \quad (i\geqq m_\nu)$$
und
$$D_{2^j}(x\dot{+}u)=0 \quad \left(x\in F\dot{+}\frac{1}{2^{i+1}},\ u\in I_{\nu'}\dot{+}\frac{1}{2^{i+1}}\dot{+}\frac{1}{2^{j+1}},\ j\geqq m_{\nu'}\right)$$
gilt, so erhalten wir auf Grund von (1.7)
$$\sum_{i=0}^{\infty}2^i\int_F \Psi_i(x)\,dx =$$
$$=\sum_{\nu=1}^{\infty}\sum_{i=0}^{m_\nu-1}2^i\sum_{I_{\nu'}\dot{+}\frac{1}{2^{i+1}}\subset I_\nu}|I_{\nu'}|\sum_{j=m_\nu}^{m_{\nu'}-1}\int_{I_{\nu'}\dot{+}\frac{1}{2^{i+1}}\dot{+}\frac{1}{2^{j+1}}} w^*(u)\left(\int_{F\dot{+}\frac{1}{2^{i+1}}\cap I_\nu} D_{2^j}(x\dot{+}u)\,dx\right)du \leqq$$
$$\leqq\sum_{\nu=1}^{\infty}\sum_{i=0}^{m_\nu-1}2^i\sum_{I_{\nu'}\dot{+}\frac{1}{2^{i+1}}\subset I_\nu}|I_{\nu'}|\int_{I_\nu} w^*(u)\,du \leqq \sum_{\nu=1}^{\infty}\left(\int_{I_\nu} w^*(u)\,du\right)\sum_{i=0}^{m_\nu-1}2^i|I_\nu| \leqq$$
$$\leqq\sum_{\nu=1}^{\infty}\int_{I_\nu} w^*(u)\,du=\int_0^1 w^*(u)\,du,$$
womit auch c) bewiesen ist.

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[3]) $F\dot{+}\frac{1}{2^{i+1}}$ bezeichnet die Menge $\left\{x\dot{+}\frac{1}{2^{i+1}};\ x\in F\right\}$.

§2. Beweis des Satzes

Es sei $M > \int\limits_0^1 |f(u)|du$ eine beliebige positive Zahl. Durch Anwendung von Hilfssatz III und auf Grund der Minkowskischen Ungleichung ergibt sich

$$h_n(f, x; 2) \leqq h_n(f_1, x; 2) + h_n(w, x; 2), \tag{2. 1}$$

wobei auf Grund von Hilfssatz III d) $h_n(f_1, x; 2) = o(1)$ (f.ü.a. in [0, 1]) besteht. (Siehe [6] oder [7].) Für $2^{k(n)-1} < 2^n \leqq 2^{k(n)}$ gilt $h_n(w, x; 2) \leqq \sqrt{2}\, h_{2^{k(n)}}(w, x; 2)$, weiterhin nach (1. 1) und (1. 2) besteht

$$h^2_{2^k}(w, x; 2) = \frac{1}{2^k} \sum_{\mu=0}^{2^k-1} \Big(\int\limits_0^1 (w(x \dotplus u) - w(x)) D_\mu(u)\, du\Big)^2 \leqq \tag{2. 2}$$

$$\leqq \frac{1}{2^k} \sum_{\mu=0}^{2^k-1} \Big(\int\limits_0^1 (w(x \dotplus u) - w(x)) D_{2^k}(u)\, du\Big)^2 + \frac{1}{2^k} \sum_{\mu=0}^{2^k-1} \Big(\int\limits_0^1 (w(x \dotplus u) - w(x)) \psi_\mu(u)\, du\Big)^2 +$$

$$+ \frac{1}{2^k} \sum_{\mu=0}^{2^k-1} \Big(\int\limits_0^1 [w(x \dotplus u) - w(x)] D^*_\mu(u; 0, k)\, du\Big)^2 =$$

$$= \big(S_{2^k}(w; x) - w(x)\big)^2 + \frac{1}{2^k} \sum_{\mu=0}^{2^k-1} [c_\mu(w)]^2 + \frac{w^2(x)}{2^k} + h^*_{2^k}(w, x; 2)$$

mit $k = k(n)$, wobei der Grenzwert der ersten drei Glieder fast überall gleich Null ist (siehe z.B. [1]). Auf Grund von (1. 1) erhalten wir

$$h^*_{2^k}(w, x; 2) = \frac{1}{2^k} \sum_{\mu=0}^{2^k-1} \Big(\int\limits_0^1 [w(x \dotplus u) - w(x)] D^*_\mu(u; 0, k)\, du\Big)^2 \leqq \tag{2. 3}$$

$$\leqq \frac{1}{2^{k-1}} \sum_{\mu=0}^{2^k-1} \left(\int\limits_0^1 [w(x \dotplus u) - w(x)] \sum_{i=0}^{l-1} r_i(u) \psi_\mu\left(u \dotplus \frac{1}{2^{i+1}}\right) D_{2^i}(u)\, du\right)^2 +$$

$$+ \frac{1}{2^{k-1}} \sum_{\mu=0}^{2^k-1} \Big(\int\limits_0^1 [w(x \dotplus u) - w(x)] D^*_\mu(u; l, k)\, du\Big)^2 \leqq$$

$$\leqq \sum_{i=0}^{l-1} \frac{2l}{2^k} \sum_{\mu=0}^{2^k-1} \Big(\int\limits_0^1 [w(x \dotplus u) - w(x)] r_i(u) D_{2^i}(u) \psi_\mu(u)\, du\Big)^2 +$$

$$+ \frac{2}{2^k} \sum_{\mu=0}^{2^k-1} \Big(\int\limits_0^1 [w(x \dotplus u) - w(x)] D^*_\mu(u; l, k)\, du\Big)^2 = 2\big(\sigma_k^{(1)}(l; x) + \sigma_k^{(2)}(l; x)\big)$$

für jede fixierte natürliche Zahl $l (\leqq k)$. Wir setzen

$$W_{x,i}(u) = [w(x \dotplus u) - w(x)] r_i(u) D_{2^i}(u) \quad (i = 1, 2, \cdots, l-1).$$

Dann gilt

$$\sigma_k^{(1)}(l;x) = l\sum_{i=0}^{l-1}\frac{1}{2^k}\sum_{\mu=0}^{2^k-1}|c_\mu(W_{x,i})|\to 0 \qquad (k\to\infty). \tag{2.4}$$

Auf Grund von (1. 5) ergibt sich

$$\sigma_k^{(2)}(l;x) = \frac{1}{2^k}\sum_{\mu=0}^{2^k-1}\Big(\int_0^1[w(x\dotplus u)-w(x)]D_\mu^*(u;l,k)\,du\Big)^2= \tag{2.5}$$

$$=\frac{1}{2^k}\sum_{\mu=0}^{2^k-1}\Big(\int_0^1[w(x\dotplus u)-w(x)]D_\mu^*(u;l,k)\,du\Big)\Big(\int_0^1[w(x\dotplus v)-w(x)]D_\mu^*(v;l,k)\,dv\Big)=$$

$$=\int_0^1\int_0^1[w(x\dotplus u)-w(x)][w(x\dotplus v)-w(x)]\frac{1}{2^k}\sum_{\mu=0}^{2^k-1}D_\mu^*(u;l,k)D_\mu^*(v;l,k)\,du\,dv=$$

$$=\int_0^1\int_0^1[w(x\dotplus u)-w(x)][w(x\dotplus v)-w(x)]K_{2^k}(u,v;l)\,du\,dv.$$

Auf Grund von (2. 1), (2. 2), (2. 3) und (2. 4) gilt

$$h_n^2(f,x;2) = O(1)\sigma_{k(n)}^{(2)}(l;x)+o(1) \qquad (n\to\infty)$$

für jede fixierte natürliche Zahl $l(\leqq k)$ f.ü.a. in [0, 1].

Da M beliebig groß gewählt werden kann, so genügt es auf Grund von Hilfssatz III e) zu zeigen, daß

$$\sigma_k^{(2)}(l;x)\leqq\varepsilon(l;x) \quad \text{für} \quad x\in F=[0,1)-\bigcup_{\nu=1}^{\infty}I_\nu \tag{2.6}$$

gilt, wobei $\varepsilon(l;x)\to 0$ $(l\to\infty)$ f.ü.a. in F.

Aus (1. 5), Hilfssatz III b) und (2. 5) mit Berücksichtigung von $w(x)=0$ $(x\in F)$ folgt:

$$\sigma_k^{(2)}(l;x)\leqq\sum_{\nu,\nu'=1}^{\infty}\frac{1}{2^k}\sum_{i,j=l}^{k-1}\cdot \tag{2.7}$$

$$\cdot\left|\int_0^1\int_0^1 w_{\nu'}(x\dotplus v)w_\nu(x\dotplus u)r_i(u)D_{2^i}(u)r_j(v)D_{2^j}(v)D_{2^k}\left(u\dotplus v\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)du\,dv\right|\leqq$$

$$\leqq 2\sum_{\nu=1}^{\infty}\sum_{m_{\nu'}\geqq m_\nu}\frac{1}{2^k}\sum_{i,j=l}^{k-1}\cdot$$

$$\cdot\left|\int_0^1\int_0^1 w_\nu(x\dotplus u)w_{\nu'}(x\dotplus v)r_i(u)D_{2^i}(u)r_j(v)D_{2^j}(v)D_{2^k}\left(u\dotplus v\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)du\,dv\right|.$$

Da nach (8)

$$D_{2^{k_1}}(u)D_{2^{k_2}}(v) = D_{2^{k_1}}(u\dotplus v)D_{2^{k_2}}(v) \qquad (k_1\leqq k_2) \tag{2.8}$$

gilt, so ist auf Grund von Hilfssatz III c)

$$\frac{1}{2^k}\left|\int_0^1 w_{\nu'}(v\dotplus x)r_j(v)D_{2^j}(v)D_{2^k}\left(u\dotplus v\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)dv\right| =$$

$$= 2^{-k}D_{2^j}\left(u\dotplus\frac{1}{2^{i+1}}\right)\left|\int_0^1 w_{\nu'}(v\dotplus x)r_j(v)D_{2^k}\left(u\dotplus v\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)dv\right| \leqq$$

$$\leqq\begin{cases}0, \text{ wenn } u\dotplus x\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\notin I_{\nu'},\\ 4M\,|I_{\nu'}|\,D_{2^j}\left(u\dotplus\frac{1}{2^{i+1}}\right), \text{ wenn } u\dotplus x\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\in I_{\nu'}.\end{cases}$$

Aus (1. 6) und (2. 7) folgt

$$\sigma_k^{(2)}(l;x)\leqq \tag{2.9}$$

$$\leqq 8M\sum_{\nu=1}^{\infty}\sum_{i,j=l}^{k-1}\int_0^1 |w_\nu(x\dotplus u)|\,D_{2^i}(u)D_{2^j}\left(u\dotplus\frac{1}{2^{i+1}}\right)\sum_{m_{\nu'}\geqq m_\nu}\varphi_{\nu'}\left(u\dotplus x\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)du =$$

$$= 8M\sum_{\nu=1}^{\infty}\sum_{i=l}^{k-1}\left(\sum_{j=l}^{i}+\sum_{j=i+1}^{k-1}\right)\int_0^1 |w_\nu(x\dotplus u)|\,D_{2^i}(u)D_{2^j}\left(u\dotplus\frac{1}{2^{i+1}}\right)\cdot$$

$$\cdot\,\Phi_{m_\nu}\left(u\dotplus x\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)du = 8M\big(\sigma_k^{(3)}(l;x)+\sigma_k^{(4)}(l;x)\big).$$

Wir bezeichnen mit $J_i(x)$ das den Punkt x enthaltende dyadische Intervall $\left[\frac{\alpha_i(x)}{2^i},\frac{\alpha_i(x)+1}{2^i}\right)$ wobei $0\leqq\alpha_i<2^i$, $\alpha_i\in G$. Dann gilt auf Grund von (8), Hilfssatz III, (1. 6), (2. 8) und (2. 9) für $x\in F$

$$\sigma_k^{(3)}(l;x)=\sum_{\nu=1}^{\infty}\sum_{i=l}^{k-1}\sum_{j=l}^{i}2^j\int_0^1 |w_\nu(x\dotplus u)|\,D_{2^i}(u)\,\Phi_{m_\nu}\left(u\dotplus x\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)du =$$

$$=\sum_{\nu=1}^{\infty}\sum_{i=l}^{k-1}\sum_{j=l}^{i}2^j\int_0^1 |w_\nu(u)|\,D_{2^i}(x\dotplus u)\,\Phi_{m_\nu}\left(u\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right)du =$$

$$=\sum_{i=l}^{k-1}\sum_{j=l}^{i}2^j\sum_{I_\nu\subset J_i(x)}\;\sum_{I_{\nu'}\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\subset I_\nu}2^i\,|I_{\nu'}|\int_{I_{\nu'}\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}}|w_\nu(u)|\,du\leqq$$

$$\leqq\sum_{i=l}^{\infty}\sum_{j=l}^{i}2^j\sum_{I_\nu\subset J_i(x)}\;\sum_{I_{\nu'}\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\subset I_\nu}2^i\,|I_{\nu'}|\int_{I_\nu}|w_\nu(u)|\,du\leqq$$

$$\leqq 4M\sum_{i=l}^{\infty}\sum_{j=l}^{i}2^j\sum_{I_\nu\subset J_i(x)}2^i|I_\nu|^2 = 4M\sum_{i=l}^{\infty}\sum_{j=l}^{i}2^j\int_0^1\Phi_0(u)D_{2^i}(x\dotplus u)\,du<$$

$$< 8M\sum_{i=l}^{\infty}2^i\int_0^1\Phi_0(u)D_{2^i}(x\dotplus u)\,du,$$

woraus sich nach Hilfssatz IV a) die Relation

$$\sigma_k^{(3)}(l;x) \leqq \varepsilon_1(l;x) \to 0 \qquad (l\to\infty), \tag{2.10}$$

f.ü.a. in F ergibt.

Aus (2. 8) und (2. 9) folgt

$$\sigma_k^{(4)}(l;x) =$$

$$= \sum_{\nu=1}^{\infty}\sum_{i=l}^{k-1}\sum_{j=i+1}^{k-1}\int_0^1 |w_\nu(x\dotplus u)|\, D_{2^i}(u) D_{2^j}\left(u\dotplus\frac{1}{2^{i+1}}\right)\Phi_{m_\nu}\left(u\dotplus x\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right) du =$$

$$= \sum_{\nu=1}^{\infty}\sum_{i=l}^{k-1} 2^i \sum_{j=i+1}^{k-1}\int_0^1 |w_\nu(u)|\, D_{2^j}\left(u\dotplus x\dotplus\frac{1}{2^{i+1}}\right)\Phi_{m_\nu}\left(u\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right) du =$$

$$= \sum_{i=l}^{k-1} 2^i \sum_{j=i+1}^{k-1}\Big(\sum_{m_\nu\leqq j}+\sum_{m_\nu>j}\Big)\int_0^1 |w_\nu(u)|\, D_{2^j}\left(u\dotplus x\dotplus\frac{1}{2^{i+1}}\right)\Phi_{m_\nu}\left(u\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right) du =$$

$$= \sigma_k^{(5)}(l;x)+\sigma_k^{(6)}(l;x).$$

Aus $D_{2^j}\left(x\dotplus u\dotplus\frac{1}{2^{i+1}}\right) = 0 \left(x\dotplus\frac{1}{2^{i+1}}\notin I_\nu,\ u\in I_\nu,\ j\geqq m_\nu\right)$ auf Grund von (1. 6) und Hilfssatz IV ergibt sich

$$\sigma_k^{(5)}(l;x) = \tag{2.11}$$

$$= \sum_{i=l}^{k-1} 2^i \sum_{j=i+1}^{k-1}\sum_{m_\nu\leqq j}\int_0^1 |w_\nu(u)|\, D_{2^j}\left(u\dotplus x\dotplus\frac{1}{2^{i+1}}\right)\Phi_{m_\nu}\left(u\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right) du \leqq$$

$$\leqq \sum_{i=l}^{k-1} 2^i \Psi_i(|w|;x) \leqq \sum_{i=l}^{\infty} 2^i \Psi_i(|w|;x) = \varepsilon_2(l;x)\to 0 \qquad (l\to\infty)$$

f. ü. a. in F.

Endlich folgt aus Hilfssatz III und IV:

$$\sigma_k^{(6)}(l;x) = \sum_{i=l}^{k-1} 2^i \sum_{j=i+1}^{k-1}\sum_{m_\nu>j}\int_0^1 |w_\nu(u)|\, D_{2^j}\left(u\dotplus x\dotplus\frac{1}{2^{i+1}}\right)\Phi_{m_\nu}\left(u\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\right) du =$$

$$= \sum_{i=l}^{k-1} 2^i \sum_{j=i+1}^{k-1}\sum_{I_\nu\subset J_j\left(x\dotplus\frac{1}{2^{i+1}}\right)}\ \sum_{I_{\nu'}\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}\subset I_\nu} 2^j |I_{\nu'}| \int_{I_{\nu'}\dotplus\frac{1}{2^{i+1}}\dotplus\frac{1}{2^{j+1}}} |w_\nu(u)|\, du \leqq$$

$$\leqq \sum_{i=l}^{\infty} 2^i \sum_{j=i+1}^{\infty}\sum_{I_\nu\subset J_j\left(x\dotplus\frac{1}{2^{i+1}}\right)} 2^j |I_\nu| \int_{I_\nu} |w_\nu(u)|\, du \leqq 4M \sum_{i=l}^{\infty} 2^i \sum_{j=i+1}^{\infty}\sum_{I_\nu\subset J_j\left(x\dotplus\frac{1}{2^{i+1}}\right)} 2^j |I_\nu|^2 \leqq$$

$$\leqq 4M \sum_{i=l}^{\infty} 2^i \sum_{j=i+1}^{\infty}\int_0^1 D_{2^j}\left(u\dotplus x\dotplus\frac{1}{2^{i+1}}\right)\Phi_j(u)\, du = \varepsilon_3(l;x)\to 0 \qquad (l\to\infty)$$

f.ü.a. in F. Wegen (2. 10) und (2. 11) folgt hieraus (2. 6), woraus sich die Behauptung unseres Satzes ergibt.

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A note on the existence of derivations

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Let K be a commutative ring with unit; $\mathfrak{A}$, $\mathfrak{B}$ two K-algebras, and denote by $\mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ the (left) K-module of all K-linear mappings of $\mathfrak{A}$ into $\mathfrak{B}$. We write $\mathfrak{L}(\mathfrak{A})$ for $\mathfrak{L}(\mathfrak{A}, \mathfrak{A})$ and note that this is a K-algebra. If φ, $D \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ and D satisfies the equation

$$D(xy) = Dx \cdot \varphi y + \varphi x \cdot Dy$$

for all $x, y \in \mathfrak{A}$, we call D a φ-derivation. If $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ and $I \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ is the identity map on $\mathfrak{A}$, an I-derivation will be termed a derivation. We will only consider φ-derivations where φ is a homomorphism.

Note that our φ-derivations are different from the φ-derivations of AMITSUR [1] and JACOBSON [3], which are additive mappings defined on fields satisfying $D(xy) = \varphi x \cdot Dy + Dx \cdot y$ (or $D(xy) = Dx \cdot \varphi y + x \cdot Dy$) for all x, y in the field. Our φ-derivations satisfy the defining equation for the (φ, φ)- derivations of [4], p. 177; they are closely related to the φ-derivations of order one of [5].

Now let $D \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ be a φ-derivation of $\mathfrak{A}$ into $\mathfrak{B}$ for some homomorphism $\varphi \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$. We define the 'spread' $\mathfrak{S}$ of the D as the smallest subalgebra of $\mathfrak{B}$ containing both the range of D and the range of φ. $\mathfrak{S}$ is in fact the smallest subalgebra of $\mathfrak{B}$ such that both φ and D are in $\mathfrak{L}(\mathfrak{A}, \mathfrak{S})$, that is, such that D is a φ-derivation from $\mathfrak{A}$ into $\mathfrak{S}$. It is clear that $\mathfrak{S}$ depends on the mapping φ, and if D is a φ-derivation for more than one φ it may have more than one spread. In the cases where we use this notion of spread however, it will be clear which mapping φ is being considered, and no explicit mention of it will be made.

We will denote the algebra direct sum of two K-algebras $\mathfrak{A}, \mathfrak{B}$ by $\mathfrak{A} \oplus \mathfrak{B}$, and their K-module direct sum by $\mathfrak{A} + \mathfrak{B}$. Thus $\mathfrak{A} \oplus \mathfrak{B}$ is the K-algebra of all pairs (a, b) with $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and componentwise operations, while $\mathfrak{A} + \mathfrak{B}$ is the K-module of all such pairs and will not be assumed to have the algebra structure of $\mathfrak{A} \oplus \mathfrak{B}$.

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If $\varphi \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ is a homomorphism, the K-linear isomorphism Φ of $\mathfrak{A}$ into $\mathfrak{A} \oplus \mathfrak{B}$ defined by

$$\Phi x = (x, \varphi x)$$

for all $x \in \mathfrak{A}$, will be called the φ-embedding of $\mathfrak{A}$ in $\mathfrak{A} \oplus \mathfrak{B}$. Any isomorphism of this type will be termed an embedding of $\mathfrak{A}$ in $\mathfrak{A} \oplus \mathfrak{B}$.

With these notations we have the following result.

Theorem 1. *Let $\mathfrak{A}$ be a K-algebra with a (K-module) direct sum decomposition $\mathfrak{A} = \mathfrak{B} + \mathfrak{J}$ where $\mathfrak{J}$ is a proper left ideal of $\mathfrak{A}$ and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ but not a left ideal. Then there is an embedding Φ of $\mathfrak{A}$ in $\mathfrak{A} \oplus \mathfrak{L}(\mathfrak{A})$, such that $\mathfrak{A}$ admits a nonzero Φ-derivation $D: \mathfrak{A} \to \mathfrak{A} \oplus \mathfrak{L}(\mathfrak{A})$ satisfying $D(\mathfrak{B}) = \{0\}$.*

Proof. For $a \in \mathfrak{A}$, let $a = x + y$ be the unique decomposition of a with $x \in \mathfrak{B}$, $y \in \mathfrak{J}$, and define idempotent mappings $P, Q \in \mathfrak{L}(\mathfrak{A})$ by $Pa = x$, $Qa = y$. Then we have immediately that $P^2 = P$, $PQ = QP = 0$, $Q^2 = Q$ and $P + Q = I$.

For $a \in \mathfrak{A}$ denote by $\tilde{a}$ the image of a in $\mathfrak{L}(\mathfrak{A})$ under the left regular representation and define K-linear mappings φ, Δ from $\mathfrak{A}$ into $\mathfrak{L}(\mathfrak{A})$ by $\varphi a = P\tilde{a}P + Q\tilde{a}Q$ and $\Delta a = Q\tilde{a}P$ for all $a \in \mathfrak{A}$. Since $\mathfrak{J}$ is a left ideal it is invariant under each $\tilde{a}$ for $a \in \mathfrak{A}$ and so $P\tilde{a}Q = 0$ for any $a \in \mathfrak{A}$. But then if $x, y \in \mathfrak{A}$, $\widetilde{(xy)} = \tilde{x}\,\tilde{y}$ and so

$$\varphi(xy) = P\tilde{x}(P+Q)\tilde{y}P + Q\tilde{x}(P+Q)\tilde{y}Q = P\tilde{x}P\tilde{y}P + Q\tilde{x}Q\tilde{y}Q =$$
$$= (P\tilde{x}P + Q\tilde{x}Q)(P\tilde{y}P + Q\tilde{y}Q) = \varphi x \cdot \varphi y.$$

Also

$$\Delta(xy) = Q\tilde{x}(P+Q)\tilde{y}P = Q\tilde{x}P\tilde{y}P + Q\tilde{x}Q\tilde{y}P =$$
$$= Q\tilde{x}P(P+Q)\tilde{y}(P+Q) + (P+Q)\tilde{x}(P+Q)Q\tilde{y}P = \Delta x \cdot \varphi y + \varphi x \cdot \Delta y.$$

Thus φ is a homomorphism and Δ is a φ-derivation, which furthermore is non-zero since $\mathfrak{B}$ is not a left ideal.

We now make use of a construction of Singer and Wermer [6]. Let Φ be the φ-embedding of $\mathfrak{A}$ in $\mathfrak{A} \oplus \mathfrak{L}(\mathfrak{A})$ and define a K-linear mapping $D: \mathfrak{A} \to \mathfrak{A} \oplus \mathfrak{L}(\mathfrak{A})$ by $Da = (0, \Delta a)$ for all $a \in \mathfrak{A}$. It is easily seen that D is a Φ-derivation, non-zero since Δ is non-zero. If $x \in \mathfrak{B}$ then $(\tilde{x}P)y \in \mathfrak{B}$ for any $y \in \mathfrak{A}$, so that $(Q\tilde{x}P)y = 0$. Thus $\Delta x = 0$ if $x \in \mathfrak{B}$, that is, $D(\mathfrak{B}) = \{0\}$.

The reason for wanting Φ to be an isomorphism is that we can identify $\mathfrak{A}$ with $\Phi(\mathfrak{A})$ to get the following result.

Corollary 1. *Let $\mathfrak{A}$ satisfy the conditions of the theorem. Then $\mathfrak{A}$ admits a non-zero derivation D into an extension of $\mathfrak{A}$ such that $D(\mathfrak{B}) = \{0\}$.*

The spread of (the φ-derivation) D is in general non-commutative even when $\mathfrak{A}$ is commutative [1]). A necessary and sufficient condition for the spread to be commutative is given by the following result.

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[1]) The range of D is easily seen to be a zero ring.

Theorem 2. *If $\mathfrak{A}$ is commutative then the spread of D is commutative if and only if* $\mathfrak{J}^2\mathfrak{B}=0$.

Proof. From the definition of D, the spread of D is commutative if and only if the spread of Δ is commutative. Now the spread of Δ is the algebra generated by the set $\{\varphi a, \Delta a: a\in\mathfrak{A}\}$. Since $\mathfrak{A}$ is commutative and φ is a homomorphism $\varphi x\cdot\varphi y=\varphi y\cdot\varphi x$ for all $x, y\in\mathfrak{A}$. Also, from the definitions, $\Delta x\cdot\Delta y=\Delta y\cdot\Delta x=0$. Thus it suffices to consider necessary and sufficient conditions for $\Delta x\cdot\varphi y=\varphi y\cdot\Delta x$, that is, $Q\tilde{x}P\tilde{y}P=Q\tilde{y}Q\tilde{x}P$ for all $x, y\in\mathfrak{A}$.

(i) Necessity of stated condition. Let $x, y\in\mathfrak{J}$, $z\in\mathfrak{B}$. Then $(Q\tilde{x}P\tilde{y}P)z=(Q\tilde{x}P\tilde{y})z=0$ since $\tilde{y}(z)=yz\in\mathfrak{J}$ and $P(\mathfrak{J})=\{0\}$. On the other hand, $(Q\tilde{y}Q\tilde{x}P)z=(Q\tilde{y}Q\tilde{x})z=(Q\tilde{y}Q)xz=(Q\tilde{y})xz=Q(yxz)=yxz$. Thus the condition is necessary.

(ii) Sufficiency of stated condition. If $z\in\mathfrak{J}$ then $(Q\tilde{x}P\tilde{y}P)z=0=(Q\tilde{y}Q\tilde{x}P)z$ for any $x, y\in\mathfrak{A}$ since $Pz=0$. Since $\mathfrak{A}=\mathfrak{B}+\mathfrak{J}$ it thus suffices to consider $z\in\mathfrak{B}$. Thus suppose $z\in\mathfrak{B}$, and let $x, y\in\mathfrak{A}$ have decompositions $x=u_1+v_1$; $y=u_2+v_2$; $u_i\in\mathfrak{B}, v_i\in\mathfrak{J}$ for $i=1, 2$. Then the decompositions of xz, yz, xu_2z are u_1z+v_1z, u_2z+v_2z, $u_1u_2z+v_1u_2z$, respectively. Thus $(Q\tilde{x}P\tilde{y}P)z=(Q\tilde{x}P)(u_2z+v_2z)=Q(xu_2z)=v_1u_2z$, and $(Q\tilde{y}Q\tilde{x}P)z=(Q\tilde{y}Q)(u_1z+v_1z)=Q(yv_1z)=yv_1z$. The difference between these is $yv_1z-v_1u_2z=v_1z(y-u_2)=v_1v_2z=0$, since $v_1, v_2\in\mathfrak{J}$, $z\in\mathfrak{B}$. Thus the condition is sufficient.

It follows that the extension of $\mathfrak{A}$ in Corollary 1 may be taken to be commutative if $\mathfrak{A}$ is commutative and $\mathfrak{J}^2\mathfrak{B}=0$.

Corollary 2. *Let $\mathfrak{A}$ be a commutative K-algebra with $\mathfrak{A}^2=\mathfrak{A}$.* [2]) *Suppose that $\mathfrak{A}$ has a direct sum decomposition $\mathfrak{A}=\mathfrak{B}+\mathfrak{N}$ where $\mathfrak{B}$ is a subalgebra and $\mathfrak{N}$ is a nontrivial nilpotent ideal. Then $\mathfrak{A}$ admits a non-zero derivation into an extension algebra of $\mathfrak{A}$ which annihilates $\mathfrak{B}$. If $\mathfrak{N}$ is a zero ring or if the sum is an algebraic direct sum then the extension algebra may be taken to be commutative.*

Proof. By hypothesis $\mathfrak{B}$ is a subalgebra, so using Corollary 1 it suffices to show that it is not an ideal. Supposing to the contrary, we have $\mathfrak{B}\mathfrak{N}\subseteqq\mathfrak{B}$ and so $\mathfrak{A}^2\subseteqq\mathfrak{B}+\mathfrak{N}^2$ whence $\mathfrak{N}^2=\mathfrak{N}$. But this is impossible since $\mathfrak{N}$ is non-trivial and nilpotent.

The last statement is clear from Theorem 2.

Acknowledgements The idea of Theorem 1 stemmed from a perusal of the matrix proof of the theorem of MASCHKE in the theory of group representations. See, for example, Theorem 16. 3. 1 of [2].

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[2]) This is true, for instance, if $\mathfrak{A}$ has an identity.

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Dini derivatives of semicontinuous functions. I

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1. In a paper [3] NEUGEBAUER has proved that for a continuous real function f, the upper and the lower derivatives of f on the right are respectively equal to the upper and the lower derivatives of f on the left, except a set of the first category. That this result is not true for arbitrary real function is also remarked in [3]. BRUCKNER and GOFFMAN [1] have shown that the result of NEUGEBAUER [3] comes as a corollary to a more general theorem. In the present paper we establish certain results analogous to the results of NEUGEBAUER [3] considering semicontinuous functions. Also some other results analogous to the results of [1] are established. Though the former results are simple consequences of the latter, we have established them independently because of their basic importance.

Throughout the paper f will denote a function from R to R where R is the set of real numbers and $D^+f(x_0)$ etc. will denote, as usual, the Dini derivatives of f at x_0.

2. For convenience let us state the following obvious

Lemma. *If $f(x)$ is upper semicontinuous and $g(x)$ is continuous at ξ, and if $g(\xi)>0$, then $\dfrac{f(x)}{g(x)}$ is upper semicontinuous at ξ.*

Theorem 1. *If $f\colon R\to R$ is upper semicontinuous, then the set*

$$\{x\colon D_-f(x)>D_+f(x)\}\cup\{x\colon D^-f(x)>D^+f(x)\}$$

is of the first category.

Proof. Let $E=\{x\colon D_-f(x)>D_+f(x)\}$. For $x\in E$, we choose a rational number r and a positive integer n such that

$$\frac{f(x-h)-f(x)}{-h}>r>D_+f(x)\quad\text{for all } h,\ 0<h<\frac{1}{n}.$$

Let E_{rn} denote the set of all points of E satisfying the above relation. Then

$$E = \bigcup_r \bigcup_n E_{rn},$$

where the union is taken over all rationals r and over all positive integers n. We shall show that E_{rn} is nondense for all r and all n.

If possible, let E_{rn} be dense in some interval (a, b). We choose two points ξ and ξ_2 such that $\xi \in (a, b) \cap E_{rn}$, $\xi_2 \in (a, b)$, $0 < \xi_2 - \xi < \frac{1}{n}$ and

$$\frac{f(\xi_2) - f(\xi)}{\xi_2 - \xi} < r.$$

By the lemma $\frac{f(x)-f(\xi)}{x-\xi}$ is upper semicontinuous at ξ_2 and so there is a $\delta > 0$ such that

$$\frac{f(x)-f(\xi)}{x-\xi} < r \quad \text{whenever } \xi_2 - \delta < x < \xi_2 + \delta.$$

We may choose δ such that $(\xi_2 - \delta, \xi_2 + \delta) \subset (a, b)$ and $\xi < \xi_2 - \delta < \xi_2 + \delta < \xi + \frac{1}{n}$. Since E_{rn} is dense in (a, b), there is a point $\xi_1 \in (\xi_2 - \delta, \xi_2 + \delta) \cap E_{rn}$. So,

(1)
$$\frac{f(\xi_1) - f(\xi)}{\xi_1 - \xi} < r,$$

Again, since $\xi_1 \in E_{rn}$ and $0 < \xi_1 - \xi < \frac{1}{n}$, we have

(2)
$$\frac{f(\xi) - f(\xi_1)}{\xi - \xi_1} > r.$$

Since (1) and (2) are contradictory, we conclude that the set E_{rn} is nondense and consequently the set E is of the first category. Similarly we can show that the set

$$F = \{x : D^- f(x) > D^+ f(x)\}$$

is of the first category.

In a similar manner we get the following theorem.

Theorem 2. *If $f: R \to R$ is lower semicontinuous then the set*

$$\{x : D_- f(x) < D_+ f(x)\} \cup \{x : D^- f(x) < D^+ f(x)\}$$

is of the first category.

We remark that the conclusion of Theorems 1 and 2 remains valid if we assume the semicontinuity of $f(x)$ in one side only.

Theorem 3. *If $f\colon R\to R$ is upper semicontinuous and r is a real number then each of the sets*

$$E=\{x\colon D^+f(x)<r\} \quad \textit{and} \quad F=\{x\colon D_-f(x)>r\}$$

is of the type F_σ.

Proof. We shall prove for the set E; the proof for F is analogous.

For any positive integer n, let

$$E_n=\left\{x\colon \frac{f(y)-f(x)}{y-x}\leqq r-\frac{1}{n}, \text{ for } x<y<x+\frac{1}{n}\right\}.$$

Then it is easy to verify that

$$E=\bigcup_n E_n.$$

To show that E is of the type F_σ we have to show that E_n is closed for each n.

Let ξ be a limit point of E_n and let $\{x_i\}$ be a sequence in E_n which converges to ξ. We take a point y_0 such that $\xi<y_0<\xi+\frac{1}{n}$. Since the sequence $\{x_i\}$ converges to ξ there is a positive integer i_0 such that

$$x_i<y_0<x_i+\frac{1}{n} \text{ whenever } i\geqq i_0.$$

Since $x_i\in E_n$, we have

$$\frac{f(y_0)-f(x_i)}{y_0-x_i}\leqq r-\frac{1}{n} \text{ when } i\geqq i_0.$$

Since by the lemma $\dfrac{f(y_0)-f(x)}{y_0-x}$ is lower semicontinuous at ξ, we conclude

$$\frac{f(y_0)-f(\xi)}{y_0-\xi}\leqq r-\frac{1}{n}.$$

Since y_0 is an arbitrary point in $\xi<y<\xi+\frac{1}{n}$, we conclude $\xi\in E_n$. Hence E_n is closed and this completes the proof.

Similarly we get

Theorem 4. *If $f\colon R\to R$ is lower semicontinuous and r is a real number then each of the sets*

$$\{x\colon D_+f(x)>r\} \quad \textit{and} \quad \{x\colon D^-f(x)<r\}$$

is of the type F_σ.

3. If φ is a real function of the two variables x and y defined and continuous for $x<y$ i.e. above the line $L\colon x-y=0$, then it is known [1] that for any given

direction θ there is a set of points $\{(\xi, \xi)\}$ residual on L[1]) such that

$$\operatorname*{Lim\,sup}_{(x,y)\to(\xi,\xi)} \varphi(x,y) = \operatorname*{Lim\,sup}_{\theta:(x,y)\to(\xi,\xi)} \varphi(x,y)$$

where $\theta: (x, y) \to (\xi, \xi)$ denotes that (x, y) tends to (ξ, ξ) along the direction θ.

If, however, φ is a lower semicontinuous function of x for every fixed value of y then also the result is true (see remark below the corollary to Theorem 1 [1]) except in the case when θ is the horizontal direction i.e. when y remains fixed. In that case we may not get such a residual set on L on which the above result is true. For, consider the function $\varphi(x, y)$, which equals 1 when y is rational, and 0 when y is irrational.

Then φ is a lower semicontinuous function of x for every fixed value of y; but for each point (ξ, ξ) on L

$$\operatorname*{Lim\,sup}_{(x,y)\to(\xi,\xi)} \varphi(x,y) = 1,$$

whereas

$$\operatorname*{Lim\,sup}_{x<\xi,\,(x,\xi)\to(\xi,\xi)} \varphi(x,y) = 0$$

whenever ξ is irrational. Now if f is an upper semicontinuous real function of a single real variable x then for any two reals x, y where $x<y$, we write

$$\varphi(x,y) = \frac{f(y)-f(x)}{y-x}.$$

Then by the lemma φ is a lower semicontinuous function of x for each fixed value of y. Considering the vertical direction we get, except a set of points (ξ, ξ) of the first category on L,

$$\operatorname*{Lim\,sup}_{x<\xi,\,x\to\xi} \varphi(x,\xi) \leqq \operatorname*{Lim\,sup}_{y>\xi,\,y\to\xi} \varphi(\xi,y)$$

i.e.

$$D^-f(\xi) \leqq D^+f(\xi).$$

Again considering the direction normal to L we get except a set of first category on L

$$\operatorname*{Lim\,sup}_{x+y=2\xi,\,(x,y)\to(\xi,\xi)} \varphi(x,y) = \operatorname*{Lim\,sup}_{y>\xi,\,y\to\xi} \varphi(\xi,y)$$

i.e. $\overline{f^{(1)}}(\xi) = D^+f(\xi)$ where $\overline{f^{(1)}}(\xi)$ is the upper symmetric derivative of f at ξ [2]. Similarly except a set of the first category on L, we have

$$D^-f(\xi) \leqq \overline{f^{(1)}}(\xi).$$

Lastly, the following results are true except a set of the first category:

$$\overline{f^{(1)}}(\xi) = \overline{f^*}(\xi), \qquad \overline{f^*}(\xi) = D^+f(\xi)$$

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[1]) I. e. whose complement is of the first category.

where $\overline{f^*}(\xi)$ denotes the upper unstraddled derivatives of f at ξ, i.e.

$$\overline{f^*}(\xi) = \limsup_{x \neq y,\ (x, y) \to (\xi, \xi)} \frac{f(x) - f(y)}{x - y}.$$

Thus we get:

Theorem. *If f is upper semicontinuous then except a set of the first category the following relations are true:*

$$D^- f(x) \leqq \overline{f^{(1)}}(x) = \overline{f^*}(x) = D^+ f(x).$$

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Some algorithms for the representation of natural numbers

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1. Let $1 = a_1 < a_2 < \cdots$ be a sequence of natural numbers. Let further $\mathscr{A}$ denote the set $\{a_n\}$.

Every natural number can be represented in the form

$$n = a_{i_1} + \cdots + a_{i_\nu} \tag{1.1}$$

where $a_{i_j} \in \mathscr{A}$, $a_{i_1} \geqq a_{i_2} \geqq \cdots \geqq a_{i_\nu}$, and a_{i_1} denotes the greatest element of $\mathscr{A}$ which does not exceed n and, in general a_{i_k} denotes the greatest element of $\mathscr{A}$ which does not exceed $n-(a_{i_1}+\cdots+a_{i_{k-1}})$ $(k=2, \cdots, \nu)$.

Let $\alpha(n)$ denote the length of this representation, i.e. $\alpha(n)=\nu$, $\alpha(0)=0$.

In this paper we study the distribution of the values $\alpha(n)$ for some special set $\mathscr{A}$. In the sections 2 and 3 we shall study the cases when the differences of the consecutive elements of $\mathscr{A}$ have a limiting distribution. In the section 4 we investigate the case when $\mathscr{A}$ consists of the square numbers.

2. Let

$$d_i = a_{i+1} - a_i \ (i=1,2,\ldots); \qquad A(x) = \sum_{a_i \leqq x} 1, \tag{2.1—2.2}$$

$$\varrho_l(x) = \sum_{\substack{a_i \leqq x \\ d_i = l}} 1 \qquad (l=1,2,\ldots). \tag{2.3}$$

Set

$$T_k(x) = \sum_{n=0}^{[x]} \alpha^k(n) \qquad (k=0,1,2,\ldots), \tag{2.4}$$

$$S(N,u) = \sum_{n=0}^{N} e^{iu\alpha(n)}; \qquad \varphi_N(u) = \frac{1}{N+1} S(N,u). \tag{2.5—2.6}$$

We shall prove

Theorem 1. *If* $n^{-1}A(n) \geqq \alpha (>0)$ *for* $n=1,2,\cdots,N$, *then* $n^{-1}T_1(n) \leqq 1/\alpha$ *for* $n=1,2,\cdots,N$.

Let us now suppose that the limits

$$\text{(2. 7)—(2. 8)}\quad \lim_{x\to\infty} x^{-1}A(x) = c(>0), \qquad \lim_{x\to\infty} x^{-1}\varrho_l(x) = \varrho_l \quad (l=1,2\ \dots)$$

exist and the relation

$$\text{(2. 9)}\qquad \sum_{l=1}^{\infty} l\varrho_l = 1$$

holds.

It is known that (2. 9) is equivalent to

$$\text{(2. 10)}\qquad \overline{\lim_{x\to\infty}}\, x^{-1} \sum_{l\geqq y} \varrho_l(x) \to 0 \qquad (y\to\infty).$$

Theorem 2. *Under the assumptions* (2. 7), (2. 8), (2. 9) *the following assertions hold:*

a) *The sequence of the characteristic functions* $\varphi_N(u)$ *tends to a limit function* $\varphi(u)$ *as* $N\to\infty$, *uniformly in* u, *and the relation*

$$\text{(2. 11)}\qquad \varphi(u) = e^{iu} \sum_{l=1}^{\infty} \varphi_{l-1}(u) l\varrho_l$$

holds.

Furthermore the limits

$$\text{(2. 12)}\qquad \lim_{N\to\infty} \frac{1}{N+1} \sum_{\substack{n\leqq N\\ \alpha(n)=l}} 1 = \tau_l \qquad (l=1,2,\dots)$$

exist and

$$\sum_{l=1}^{\infty} \tau_l = 1.$$

b) *We have*

$$\text{(2. 13)}\qquad \lim_{x\to\infty} x^{-1} T_k(x) = 1 + \sum_{\nu=1}^{k} \binom{k}{\nu} \sum_{l=1}^{\infty} T_\nu(l-1)\varrho_l,$$

for $k=1,2,\cdots$, *the sum on the right hand side of* (2. 13) *being convergent.*

3. For the proof of Theorem 1 we use induction on n. Since $1\in\mathscr{A}$, so $T_1(1)/1\leqq 1/\alpha$ evidently holds. Suppose now, that $m^{-1}T_1(m)\leqq 1/\alpha$ for $m=1,\cdots,n-1$, where $1<n\leqq N$. Hence we deduce that $n^{-1}T_1(n)\leqq 1/\alpha$. Indeed we have

$$T_1(n) = \sum_{m\leqq n} \alpha(m) = \sum_{j=2}^{\nu} \sum_{a_{j-1}\leqq m<a_j} \alpha(m) + \sum_{a_\nu\leqq m\leqq n} \alpha(m),$$

where $a_\nu \leqq n < a_{\nu+1}$. Since

$$\sum_{a_{j-1}\leqq m<a_j} \alpha(m) = d_{j-1} + \sum_{\nu=0}^{d_{j-1}-1} \alpha(\nu) = d_{j-1} + T_1(d_{j-1}-1)$$

so we get

$$(3.1)\qquad T_1(n) = n + \sum_{j=2}^{\nu} T_1(d_{j-1}-1) + T_1(n-a_\nu) =$$

$$= n + \sum_{d=1}^{\infty} T_1(d-1)\varrho_d(a_{\nu-1}) + T_1(n-a_\nu).$$

If $n^{-1}T_1(n) \leqq \max\limits_{m\leqq n-1} m^{-1}T_1(m)$, then $n^{-1}T_1(n)\leqq 1/\alpha$ evidently holds. Let us now suppose that

$$n^{-1}T_1(n) = \max_{1\leqq m\leqq n} m^{-1}T_1(m).$$

Then from (3. 1) it follows that

$$\frac{T_1(n)}{n} \leqq 1 + \frac{T_1(n)}{n}\cdot\frac{1}{n}\left\{\sum_d \varrho_d(a_{\nu-1})(d-1) + (n-a_\nu)\right\} =$$

$$= 1 + \frac{T_1(n)}{n}\cdot\frac{1}{n}\left\{\sum_{j=1}^{\nu-1}(a_{j+1}-a_j-1) + (n-a_\nu)\right\} =$$

$$= 1 + \frac{T_1(n)}{n}\cdot\frac{1}{n}\{n-A(n)\} = 1 + \frac{T_1(n)}{n}\left(1-\frac{A(n)}{n}\right),$$

and consequently $\dfrac{T_1(n)}{n}\cdot\dfrac{A(n)}{n} \leqq 1$, i. e. $\dfrac{T_1(n)}{n} \leqq \dfrac{1}{\alpha}$ holds.

We begin the proof of Theorem 2. Let $a_\nu \leqq N < a_{\nu+1}$.

a) We have

$$S(N,u) = e^{iu\alpha(0)} + \sum_{j=2}^{\nu}\sum_{a_{j-1}\leqq n<a_j} e^{iu\alpha(n)} + \sum_{a_\nu\leqq n<N} e^{iu\alpha(n)} =$$

$$= 1 + e^{iu}\sum_{j=2}^{\nu} S(d_{j-1},u) + e^{iu}S(N-a_\nu,u) =$$

$$= 1 + e^{iu}\sum_{d=2}^{\infty}\frac{S(d-1,u)}{d}\, d\varrho_d(a_{\nu-1}) + e^{iu}S(N-a_\nu,u).$$

Since the limit $\lim\limits_{x\to\infty} x^{-1}A(x) = c$ exists, so $d_i = o(a_i)$ $(i\to\infty)$, and consequently $|S(N-a_\nu,u)|/N \to 0$. Hence it follows that

$$\varphi_N(u) = \frac{1}{N+1}\,\frac{a_{\nu-1}+1}{N+1}\, e^{iu}\sum_{d=2}^{\infty}\varphi_{d-1}(u)\, d\,\frac{\varrho_d(a_{\nu-1})}{a_{\nu-1}+1} + o(1).$$

Let now $\varphi(u)$ be defined by the relation (2. 11). Then

$$|\varphi_N(u)-\varphi(u)| = (1+o(1))\left|\sum_{d=2}^{\infty}\varphi_{d-1}(u)\left(\frac{d\varrho_d(a_{\nu-1})}{a_{\nu-1}+1}-d\varrho_d\right)\right|+o(1) \leqq$$

$$\leqq 2\sum_{d=2}^{\infty}\left|\frac{d\varrho_d(a_{\nu-1})}{a_{\nu-1}+1}-d\varrho_d\right|+o(1).$$

From (2. 8), (2. 9) it follows that the last sum tends to zero as $N\to\infty$, independently from u.

From (2. 11) it follows that $\varphi(u)$ is a characteristic function. Since $\varphi_N(u)$ and consequently $\varphi(u)$ are periodic functions mod 2π, so $\varphi(u)$ has a Fourier expansion

$$\varphi(u) = \sum_{n=-\infty}^{\infty}\delta_n e^{inu}.$$

Using the uniform convergence of $\varphi_N(u)$ to $\varphi(u)$ we have

$$\delta_l = \frac{1}{2\pi}\int_0^{2\pi}\varphi(u)e^{-ilu}du = \lim_{N\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\varphi_N(u)e^{-ilu}du =$$

$$= \begin{cases}\lim\limits_{N\to\infty}\dfrac{1}{N+1}\sum\limits_{\substack{n\leqq N\\ \alpha(n)=l}}1 = \tau_l & \text{for } l=1,2,\cdots,\\ 0 \quad \text{for} \quad l=0,-1,-2,\cdots.\end{cases}$$

Furthermore

$$\sum\tau_l = \sum\delta_l = \varphi(0) = 1.$$

b) We have

$$T_k(x) = \sum_{n\leqq x}\alpha^k(n) = \sum_{a_i\leqq x}\sum_{j=0}^{d_i-1}{}'(\alpha(j)+1)^k,$$

where the dash means that for $a_i\leqq x<a_{i+1}$ we sum over those j for which $j\leqq x-a_i$. Hence it follows that

$$T_k(x) = \sum_{a_i\leqq x}\sum_{j=0}^{d_i-1}\binom{k}{\nu}\alpha^\nu(j) = \sum_{\nu=0}^{k}\binom{k}{\nu}\sum_{a_i\leqq x}{}'T_\nu(d_i-1) = \sum_{\nu=0}^{k}\binom{k}{\nu}\sum_{l=1}^{\infty}{}'T_\nu(l-1)\varrho_l(x).$$

The fulfilment of the relation (2. 13) would follow from the boundedness of the sums $T_\nu(l)/l$ $(l=1,2,\cdots;\ \nu=1,\cdots,k)$ by (2. 10) immediately. The boundedness of $T_1(x)/x$ follows from Theorem 1. The proof of the general case is similar and so it can be omitted.

4. Let $\mathscr{A}$ be the set of square numbers. Introduce the notation $\log_2 x=\log\log x$ where the base of the logarithm is 2.

It is easy to prove that

$$\alpha(n)\leqq\log_2 n+5. \tag{4. 1}$$

Indeed, if

$$A(x) = \max_{n \leqq x} \alpha(n),$$

then from the inequality $n - [\sqrt{n}]^2 \leqq 2\sqrt{n}$ it follows that

$$A(x) \leqq 1 + A(2\sqrt{x}).$$

Iterating this inequality k times we have

$$A(x) \leqq k + A(2^{1+\frac{1}{2}+\cdots+1/2^{k-1}} x^{\frac{1}{2}k}) \leqq k + A(4x^{\frac{1}{2}k}).$$

Let k be the smallest integer for which $x^{\frac{1}{2}k} \leqq 2$, i.e. $k = [\log_2 x] + 1$. Since $A(8) = 4$ we have

$$A(x) \leqq \log_2 x + 5.$$

Set

$$T_k(x) = \sum_{n \leqq x} \alpha^k(n) \quad \text{and} \quad \Delta_k(x) = \sum_{n \leqq x} |\alpha(n) - \log_2 x|^k.$$

Theorem 3. *We have*

$$T_k(x) = x(\log_2 x)^k + O(x(\log_2 x)^{k-1}), \tag{4.2}$$

$$\Delta_k(x) = O(x), \tag{4.3}$$

where the constants in the O terms depend on k only.

Proof. It is evident that (4. 2) follows from (4. 3). For the proof of (4. 3) we use induction on k. The relation holds for $k=0$. Let now suppose that (4. 3) holds for $k = 0, 1, \cdots, K-1$. Then we deduce the inequality (4. 3) for $k = K$.

We have

$$\Delta_K(N) \leqq \sum_{\nu^2 \leqq N} \sum_{\nu^2 \leqq n < (\nu+1)^2} |\alpha(n) - \log_2 N|^K = \sum_{\nu^2 \leqq N} \sum_{j=0}^{2\nu} |\alpha(j) + 1 - \log_2 N|^K = \sum_{\nu \leqq \sqrt{N}} B_\nu.$$

Using the inequality

$$|a+b|^K \leqq |a|^K + \sum_{l=1}^{K} \binom{K}{l} |a|^{K-l} |b|^l$$

and consequently that

$$|\alpha(j) + 1 - \log_2 N|^K \leqq |\alpha(j) - \log_2 2\nu|^K + \sum_{l=1}^{K} \binom{K}{l} |\alpha(j) - \log_2 2\nu|^{K-l} \left| \log \frac{\log 2\nu}{\log \sqrt{N}} \right|^l$$

we obtain

$$B_\nu \leqq \Delta_K(2\nu) + \sum_{l=1}^{K} \binom{K}{l} \Delta_{K-l}(2\nu) \left| \log \frac{\log \sqrt{N}}{\log 2\nu} \right|^l.$$

Using our assumption that $\Delta_k(2\nu)\ll 2\nu$ for $k\leqq K-1$ we get

$$\Delta_K(N)\leqq\sum_{\nu\leqq\sqrt{N}}\Delta_K(2\nu)+O(\textstyle\sum_1),$$

where

$$\sum\nolimits_1\ll\sum_{\nu\leqq\sqrt{N}}\nu\left\{\left|\log\frac{\log\sqrt{N}}{\log 2\nu}\right|+\left|\log\frac{\log\sqrt{N}}{\log 2\nu}\right|^K\right\}.$$

Dividing the interval of summation $[1,\sqrt{N}]$ into subintervals of type $\left[\frac{\sqrt{N}}{2^{j+1}},\frac{\sqrt{N}}{2^j}\right]$ we easily obtain the inequality

$$\sum\nolimits_1\ll\sum_{2^j\leqq\sqrt{N}}\left(\frac{\sqrt{N}}{2^j}\right)^2\left\{\frac{j}{\log\sqrt{N}}+\left(\frac{j}{\log\sqrt{N}}\right)^K\right\}\ll\frac{N}{\log N}.$$

Hence

$$\Delta_K(N)\leqq\sum_{\nu\leqq\sqrt{N}}\Delta_K(2\nu)+O\left(\frac{N}{\log N}\right)\tag{4.4}$$

follows.

Introduce now the notation

$$\Delta_K(N)=\varepsilon(N)N.$$

We prove that $\varepsilon(N)$ is bounded; hence the inequality (4.3) follows for $k=K$, and this will finish the proof of our theorem.

Let

$$\beta_j=\max_{2^{j-1}\leqq m\leqq 2^j}\varepsilon(m)\qquad(j=1,2,\ldots).$$

From (4.4)

$$\varepsilon(N)\leqq\frac{1}{N}\sum_{\nu\leqq\sqrt{N}}\varepsilon(2\nu)2\nu+c/\log N$$

follows with a suitable constant c. Hence

$$\beta_{2l}\leqq\max_{j\leqq l+2}\beta_j+\frac{c}{l};\qquad\beta_{2l+1}\leqq\max_{j\leqq l+2}\beta_j+\frac{c}{l}.$$

Define the non-decreasing sequence of positive numbers $\gamma_4,\gamma_5,\ldots$ as follows: Let

$$\gamma_4=\gamma_5=\max(\beta_1,\beta_2)+\frac{c}{2};\qquad\gamma_{2l}=\gamma_{2l+1}=\max_{j\leqq l+2}\gamma_j+\frac{c}{l}\quad(l=3,\ldots).\tag{4.5}$$

Clearly, $\beta_j\leqq\gamma_j$ for $j\geqq 4$. So it is enough to prove that γ_n is bounded. Let

$$B(x)=\sum_{j\leqq x}\gamma_j.$$

From (4.5) it follows that

$$B(2x)\leqq 2B(x)+2\gamma_{[x]}+c\log x.$$

Furthermore from (4. 1) we can easily see that $\varepsilon(N) \ll (\log_2 N)^K$. Hence $\beta_j \ll (\log j)^K$, and so

$$\gamma_{[x]} \ll \beta_{[x]} + \log x \ll (\log x)^K$$

follows.

Set $\varphi(x) = \dfrac{B(x)}{x}$. Then $\varphi(2x) \leqq \varphi(x) + c_1 \dfrac{(\log x)^K}{x}$. So the sequence $\varphi(2^m)$ $(m=1, 2, \cdots)$ is bounded. Hence $B(x) < cx$ follows for every x. Since $\{\gamma_n\}$ is non-decreasing we have

$$\gamma_l \leqq \frac{\gamma_{l+1} + \cdots + \gamma_{2l}}{l} \leqq \frac{B_{2l}}{l} < 2c,$$

i.e. γ_l is bounded.

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Cardinals inaccessible with respect to a function defined on pairs of cardinals

By G. FODOR and A. MÁTÉ in Szeged

In the present paper we are going to prove a lemma based on the theory of stationary classes with the aid of which a formula can be derived for the cofinality number of an arbitrary cardinal. Replacing the particular function occurring on the right hand side of this formula by a function variable we are led to a generalization of cofinality and thus at last we shall get a generalization of algebraic type of the notion of inaccessible cardinals. A simple by-product of our investigations will be that in a sense almost every weakly inaccessible cardinal is strongly inaccessible too. Our lemma might be of some interest in itself as well.

Notation. In the sequel Greek letters will always denote cardinal numbers and the class of all cardinals will be denoted by C. The least ordinal exceeding a class H of cardinal numbers will be denoted by $\sup H$. This is a cardinal number unless H is a proper class; in this latter case $\sup H = On$, On denoting the class of all ordinal numbers.

1. Stationary classes[1])

Here we give a brief sketch of the most important results in the theory of stationary sets used below. We do not deal with the generalized form of the theory as given by G. FODOR and A. HAJNAL [1]; however this theory might be most illuminating in the understanding of the special theory as well.

Where the adjective "closed" if used for a subclass of C is meant in the topology induced by the natural ordering of C, a subclass of C is said to be stationary if it meets every closed proper class contained in C. One of the most important results for stationary classes is the following one (see [2], Hilfssatz).

Theorem 1. 1. *Suppose that $\{S_\alpha\}_{\alpha \in H}$ $(H \subseteq C)$ is a sequence of non-empty and non-stationary classes and the class $\{\sigma_\alpha\}_{\alpha \in H}$ of their first elements, which are*

[1]) A more detailed account of the subject presented here will be given in the authors' forthcoming book on stationary classes, regressive functions and their applications.

assumed to be mutually distinct, is not stationary either. Then the union class $\bigcup_{\alpha \in H} S_\alpha$ *is also non-stationary.*

However for later use it seems preferable to formulate this results in terms of a regressive function by which we mean a mapping f of a subclass of C into C statisfying $f(\alpha) < \alpha$. The equivalence of the next result to Theorem 1. 1 is rather obvious (see [2], Satz 2).

Theorem 1. 2. *If the regressive function f is defined on a stationary class then there exists a cardinal* μ *such that the class* $\{\xi : f(\xi) = \mu\}$, ξ *running over the domain of f, is stationary too.*

Now we derive from this last result a corollary which is already of special interest in order to achieve the proof of our main lemma mentioned below.

Corollary 1. 3. *Let* $h(\lambda, \xi)$ *an arbitrary mapping of* $C \times C$ *into* C. *Then the class*

$$S = \{\alpha : \exists (\lambda, \xi)(\lambda, \zeta < \alpha \,.\, \& \,.\, h(\lambda, \xi) \geqq \alpha)\}$$

is not stationary.

Proof. Assuming S to be stationary, by the previous theorem we obtain the existence of a cardinal λ_0 and of a stationary subclass S' of S such that for any $\alpha \in S'$ we have $\lambda_0 < \alpha$ and

$$(\exists \xi)(\xi < \alpha \,.\, \& \,.\, h(\lambda_0, \xi) > \alpha);$$

so by a repeated application of the preceding theorem we have that there exists a cardinal ξ_0 and a stationary class $S'' \subseteqq S'$ such that $\alpha \in S''$ implies $\xi_0 < \alpha$ and $h(\lambda, \xi_0) \geqq \alpha$; so S'' is not cofinal to C in contradiction to its stationarity.

2. The main lemma

In the sequel $h(\alpha, \xi)$ denotes a mapping of $C \times C$ into C satisfying

$$\sup_\alpha h(\alpha, \xi) = On \tag{2. 1}$$

whichever the cardinal ξ may be.

We start by proving the following

Lemma 4. 1. *Define the classes*

$$P(\alpha) = \{\xi : \xi < \alpha \,.\, \& \,.\, h(\alpha, \xi) > \alpha\},$$

$$Q(\alpha) = \{\xi : \xi < \alpha \,.\, \& \,.\, h(\alpha, \xi) > \sup_{\eta < \alpha} h(\eta, \xi)\},$$

depending on the arbitrary cardinal α. *Then we have* $P(\alpha) = Q(\alpha)$ *for almost all* α, *meaning by this latter expression that the exceptional* α'*s form a non-stationary class.*

Proof. (i) Assume that

$$P(\alpha) \ni \xi \notin Q(\alpha).$$

Then $\xi < \alpha$ and $\alpha < h(\alpha, \xi) \leqq \sup_{\eta < \alpha} h(\eta, \xi)$, i.e. for some $\eta < \alpha$ we have $h(\eta, \xi) > \alpha$. Because of the inequality $\eta, \xi < \alpha$ the class of α's of this kind is not stationary, according to Corollary 1. 3.

(ii) Suppose that for a stationary class S of α's we have

$$P(\alpha) \not\ni \xi_\alpha \in Q(\alpha).$$

Since $\xi_\alpha < \alpha$, by Theorem 3. 4 we obtain the existence of a cardinal ξ_0 and of a stationary class $S' \subseteqq S$ such that for $\alpha \in S'$ we have $\xi_\alpha = \xi_0$. So

$$\sup_{\eta < \alpha} h(\eta, \xi_0) < h(\alpha, \xi_0) \leqq \alpha.$$

Thus there exists a β_0 and a stationary subclass S'' of S' such that for $\alpha \in S''$ we have

$$\sup_{\eta < \alpha} h(\eta, \xi_0) = \beta_0.$$

Since S'' is stationary it is cofinal to C, so our latest equality implies

$$\sup_{\eta} h(\eta, \xi_0) = \beta_0 < On,$$

contradicting (2. 1).

Below the notation $\min H$ will indicate the first element of the class H. In case the class H is empty, then in a natural way we put $\min H = On$. The next lemma, which we might call our main lemma, can be easily derived from our preceding lemma, so the proof will not be carried out.

Lemma 2. 2. *Let*

$$p(\alpha) = \min \{\xi : h(\alpha, \xi) > \alpha\},$$

$$q(\alpha) = \min \{\xi : h(\alpha, \xi) > \sup_{\eta < \alpha} h(\eta, \alpha)\}.$$

Then for almost all α either $p(\alpha) = q(\alpha)$ or $p(\alpha), q(\alpha) \geqq \alpha$ holds.

Corollary 2. 3. *If α^* denotes the least cardinal cofinal to α, then*

$$\alpha^* = \min \{\xi : \alpha^\xi > \alpha\}$$

holds for almost all α.

Proof. As is easy to derive from a classical result of J. König [3], we have $\alpha^{\alpha^*} > \alpha$. Thus putting $h(\alpha, \xi) = \alpha^\xi$, we obtain

$$p(\alpha) \leqq \alpha^*. \tag{2. 2}$$

On the other hand we have

$$q(\alpha) \geqq \alpha^*, \tag{2. 3}$$

as a consequence of the almost obvious equality

$$\alpha^\xi = \sup_{\eta<\alpha^*} \eta^\xi,$$

being valid for every $\xi<\alpha^*$.

Making use of Lemma 2. 2 and the inequalities (2. 2) and (2. 3) we obtain that $p(\alpha)=\alpha^*$ holds for almost all α, which was to be proved.

3. Inaccessible cardinals

Now we recall two well-known definitions and will indicate heuristically the way which leads us to their generalizations.

Definition 3. 1. The cardinal number α is weakly inaccessible if it is regular (i. e. $\alpha^*=\alpha$) and moreover $\xi^+<\alpha$ for each $\xi<\alpha$, ξ^+ denoting the least cardinal number exceeding ξ. Thus denoting by I the class of all weakly inaccessible cardinals, our definition may be written in a more formal way:

$$I = \{\alpha : \alpha^* = \alpha . \& . (\forall \xi)(\xi<\alpha . \rightarrow . \xi^+<\alpha)\}.$$

Definition 3. 2. The cardinal number α is strongly inaccessible, otherwise said $\alpha\in J$, if $\alpha\in I$ and for any $\lambda, \xi<\alpha$ the inequality $\lambda^\xi<\alpha$ holds. Formally

$$J = \{\alpha\in I : (\forall \lambda, \xi)(\lambda, \xi<\alpha . \rightarrow . \lambda^\xi<\alpha)\}.$$

According to Corollary 2. 3 we can replace α^* by $\min\{\xi : \alpha^\xi \leqq \alpha\}$ for almost all α. Then the condition $\alpha^*=\alpha$ in Definition 3. 1 turns into the one

$$(\forall \xi)(\xi<\alpha . \rightarrow . \alpha^\xi \leqq \alpha).$$

So if we define the class

$$I' = \{\alpha : (\forall \xi)(\xi<\alpha . \rightarrow . \alpha^{\xi^+} \leqq \alpha\},$$

it is easily seen that the classes I' and I are almost equal, i. e. their symmetrical difference $I' \triangle I$ is not stationary. Thus it is obvious that for

$$J' = \{\alpha\in I' : (\forall \lambda, \xi)(\lambda, \xi<\alpha . \rightarrow . \lambda^\xi<\alpha)\} = \{\alpha\in I' : (\forall \lambda, \xi)(\lambda, \xi<\alpha . \rightarrow . \lambda^{\xi^+}<\alpha)\}$$

the class $J' \triangle J$ is not stationary either.

As seen in the definitions of I' and J' the function λ^{ξ^+} is crucial there which, replaced by an arbitrary mapping $h(\lambda, \xi)$ of $C\times C$ into C, allows us to generalize the above concepts in a suitable way:

Definition 3. 3. $\alpha\in I_h$ i. e. α is weakly h-inaccessible if $h(\alpha, \xi)\leqq\alpha$ for each $\xi<\alpha$. Formally $I_h = \{\alpha : (\forall \xi)(\xi<\alpha . \rightarrow . h(\alpha, \xi)\leqq\alpha)\}$.

Definition 3. 4. $\alpha\in J_h$ i. e. α is strongly h-inaccessible, if $\alpha\in I_h$ and for each $\lambda, \xi<\alpha$ we have $h(\lambda, \xi)<\alpha$. Formally $J_h = \{\alpha\in I_h : (\forall \lambda, \xi)(\lambda, \xi<\alpha . \rightarrow . h(\lambda, \xi)<\alpha)\}$.

These two definitions make it clear that there is no essential difference between weakly and strongly inaccessible cardinals. More precisely, on account of Corollary 1.3 we have

Theorem 3. 5. *The class* $I_h - J_h$ *is not stationary.*

Restating this result in the particular case $h(\alpha, \xi) = \alpha^{\xi+}$ and taking into consideration that the symmetrical differences $I' \triangle I$ and $J' \triangle J$ are non-stationary, we get that the class $I - J$ is not so either, i. e. *almost all weakly inaccessible cardinals are strongly inaccessible too.*

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A generalization of the Rees theorem in semigroups

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1. Introduction and summary

The Rees theorem asserts that a semigroup S is completely 0-simple if and only if S is isomorphic to a regular Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ over a group with zero G^0 (3. 5, [1]; see also the original paper of REES [5]). As with the Rees matrix semigroups over a group with zero, we can construct a semigroup $\mathcal{M}^0(D; I, \Lambda; P)$ starting with any semigroup D instead of a group. A natural way of generalizing the Rees theorem consists on solving the following problem: to give an abstract characterization of semigroups $\mathcal{M}^0(D; I, \Lambda; P)$, where D is taken in a class of semigroups containing the class of groups. The purpose of this paper is to give several solutions of this problem, with some restrictions on P, using the notion of a 0-matrix decomposition of a semigroup [3].

We say that a semigroup S has a 0-matrix decomposition if S has a zero 0 and there exists a congruence ϱ on S such that (a) 0 is a ϱ-class and (b) S/P is a rectangular 0-band (i.e., a completely 0-simple semigroup with trivial subgroups). In such a case, $ABA \subseteqq A$ or 0 for all ϱ-classes A, B. If all the ϱ-classes which are subsemigroups of S belong to a class $\mathcal{T}$ of semigroups, we say that S is a 0-matrix of semigroups of type $\mathcal{T}$. In case S has no zero, obvious modifications of the preceding definitions yield the concepts of a matrix decomposition and a matrix of semigroups of type $\mathcal{T}$. Using this terminology and separating the cases with and without zero, we obtain the following weakened versions of the Rees theorem:

(i) A semigroup S is a matrix of groups if and only if $S \cong \mathcal{M}(G; I, \Lambda; P)$, where G is a group (Theorem 12, [4]).

(ii) A semigroup S is a 0-matrix of groups such that the classes of the corresponding congruence ϱ satisfy $ABA = A$ or 0, if and only if $S \cong \mathcal{M}^0(G; I, \Lambda; P)$, where G is a group (4. 5 [3]).

In view of this situation, for the case without zero, we introduce the class of composable semigroups (2. 1), which, e.g., contains the class of bisimple semigroups with identity (2. 3). For the case with zero, we introduce a special kind of matrix decomposition, the Rees 0-composition (3. 3). Our main results are:

(1) A semigroup S is a matrix of composable semigroups if and only if $S \cong \mathcal{M}(D; I, \Lambda; P)$, where D is a composable semigroup and P is a $\Lambda \times I$-matrix over G, the group of units of D (3. 10).

(2) A semigroup S is a Rees 0-composition if and only if $S \cong \mathcal{M}^0(D; I, \Lambda; P)$ where D is a semigroup with identity and P is a regular $\Lambda \times I$-matrix over G^0, G being the group of units of D (3. 4).

(3) We give an abstract characterization of $\mathcal{M}^0(D; I, \Lambda; P)$ when D is a bisimple inverse semigroup with identity, P is a regular $\Lambda \times I$-matrix over G, and G is the group of units of D (5. 1). This characterization uses properties of the partially ordered set of idempotents and the fact that principal right [left] ideals form a semilattice under intersection.

In section 2, we study right [left] composable semigroups. Using the notion of an r-composition of semigroups (2. 4), introduced by YOSHIDA [8], we show in 2. 5 that any r-composition of right composable semigroups is isomorphic to $D \times R$, where D is right composable and R is a right zero semigroup. Our main results (1) and (2) are established in section 3. We also prove that the class of composable semigroups is the largest class $\mathscr{C}$ with the property that every matrix of semigroups of type $\mathscr{C}$ is a Rees composition (3. 9). Section 4 is devoted to 0-restricted homomorphisms of semigroups $\mathcal{M}^0(D; I, \Lambda; P)$ discussed above; they can be described in essentially the same manner as those of a Rees matrix semigroup over a group with zero (4. 1, 4.2). We also characterize Rees matrix semigroups which can be expressed as products of some special semigroups (4. 3). The abstract characterization described in (3) above is given in section 5. It is of interest to note that 5. 1 makes it possible to construct certain 0-bisimple regular semigroups from bisimple inverse semigroups with identity. The characterization given in 5. 1 simplifies if we assume that the idempotents form a subsemigroup. We obtain, e.g., the structure of any bisimple semigroup whose idempotents form a subsemigroup isomorphic to the Cartesian product of a semilattice with identity and a rectangular band (5. 6). In Section 6, we conclude by giving an example of a bisimple regular semigroup whose set of idempotents does not satisfy the conditions of the version of 5. 1 without zero.

Recently STEINFELD [7] gave an abstract characterization of matrix semigroups $\mathcal{M}^0(D; I, \Lambda; P)$ which are locally regular (i.e. the entries of P are not necessarily taken in G^0, where G is the group of units of D, but certain entries of P have invertibility properties). Our results concern the instance in which the entries of P are in G^0 and widely supplement those of Steinfeld in this case.

Except for the concepts defined in the paper, we follow the notation and terminology of CLIFFORD and PRESTON [1]. In section 3 and 5, we use a number of concepts introduced and results proved in [3]; however, the knowledge of [3] is not indispensable. In order to avoid repetition, instead of "S is a semigroup with identity [zero]" we write $S=S^1$ $[S=S^0]$. If $S=S^1$ $[S=S^0]$, then 1 [0] denotes the identity [zero] of S unless stated otherwise.

2. Composable semigroups

Definition 2. 1. A semigroup $S=S^1$ is called *right* [*left*] *composable* if for any $a\in S$, $axa=xa$ $[axa=ax]$ for all $x\in S$ implies $a=1$. A semigroup is called *composable* if it is both right and left composable.

The reason for this terminology as well as the importance of such semigroups will become clear later (2. 5). We consider now some properties and examples of these semigroups.

Proposition 2. 2. *A semigroup S is* [*right*] *composable if and only if $S=S^1$ and the identity transformation on S is the only inner* [*right*] *translation of S which is also a homomorphism.*

Proof. The bracketted part follows directly from the equivalence of the statements: (i) ϱ_a is a homomorphism, (ii) $(xa)(ya)=(xy)a$ for all $x, y\in S$, (iii) $aya=ya$ for all $y\in S$, when $S=S^1$.

Proposition 2. 3. *Any bisimple semigroup $S=S^1$ is composable.*

Proof. Let $a\in S$ and suppose that $axa=xa$ for all $x\in S$. Since S is bisimple, there is $z\in S$ such that $a\mathcal{L}z$ and $z\mathcal{R}1$; $a\mathcal{L}z$ implies $za=z$ since $a^2=a$. If z' is an inverse of z, then $z\mathcal{L}z'z$, which implies $z'z=z'za$. Since $axa=xa$ for all $x\in S$, we obtain $z'z=z'za=z'aza=z'az$. On the other hand, $z\mathcal{R}1$ implies $zz'=1$, which together with $z'z=z'az$ yields

$$1=zz'=z(z'z)z'=z(z'az)z'=(zz')a(zz')=1a1=a.$$

Hence S is right composable; analogously S is also left composable.

Example 1.*) Let S be a left group which is not a group. Then S^1 is right composable. Since every idempotent e of S is a right identity of S, we have $exe=ex$ for all $x\in S^1$; hence S^1 is not left composable.

Example 2. Let $S=S^1$ and let S have a minimal two-sided completely simple ideal K which is neither a left nor a right group. Further suppose that 1 is the only idempotent of S not contained in K. Then S is composable. For if $axa=xa$

*) The referee points out that the right composable semigroups are precisely those semigroups with identity containing no proper left ideals with identity (the verification is left to the reader). Hence any left simple semigroup with identity, or a semigroup S^1 where S is a left simple semigroup without identity, is right composable.

for every $x \in S$ and some $a \in S$, then $a^2 = a$ and thus either $a = 1$ or $a \in K$. The latter possibility is excluded since $axa = xa$ for every $x \in K$ implies $a\mathcal{R}x$ for every $x \in K$, which in turn implies that K is a right group, contradicting the hypothesis. Thus S is right composable; by symmetry S is also left composable.

Example 3. Let $S = S^1$ be the union of groups such that no $\mathcal{D}$-class of S different from the $\mathcal{D}$-class containing the identity is a left or a right group. Similar reasoning as in the previous example shows that S is composable.

Definition 2.4 (cf. [8]). A semigroup S is said to be an *r-composition [l-composition] of semigroups* $\{D_\lambda\}_{\lambda \in \Lambda}$ if $S = \bigcup_{\lambda \in \Lambda} D_\lambda$, $D_\lambda \cap D_\mu = \square$ if $\lambda \neq \mu$, and each D_λ is a left [right] ideal of S.

Note that if S is an r-composition of semigroups D_λ, the equivalence relation induced on S is a congruence ϱ such that S/ϱ is a right zero semigroup, and conversely, every such congruence induces an r-composition of S. Furthermore, for a given family of pairwise disjoint semigroups, there may exist no r-composition (see [8]). The importance of the class of right composable semigroups stems from the next two theorems.

Theorem 2.5. *Let S be an r-composition of right composable semigroups D_λ, $\lambda \in \Lambda$, with identities 1_λ. Then the set $R_\Lambda = \{1_\lambda | \lambda \in \Lambda\}$ is a right zero semigroup, all D_λ are isomorphic, and $S \cong D_1 \times R_\Lambda$, where D_1 is any of the semigroups D_λ.*

Proof. For any $\lambda, \mu \in \Lambda$ and $x \in D_\mu$, we get $x1_\lambda \in D_\lambda$ so that $x1_\lambda = 1_\lambda x 1_\lambda$; since also $x = 1_\mu x$, we obtain $1_\lambda 1_\mu x 1_\lambda 1_\mu = x 1_\lambda 1_\mu$ for every $x \in D_\mu$. Since $1_\lambda 1_\mu \in D_\mu$ and D_μ is right composable, it follows that $1_\lambda 1_\mu = 1_\mu$. This proves that R_Λ is a right zero semigroup. Fix any index, say $1 \in \Lambda$, and define φ by $x\varphi = (x1_1, 1_\lambda)$ if $x \in D_\lambda$. A straightforward calculation shows that φ is an isomorphism of S onto $D_1 \times R_\Lambda$. (This is a special case of Theorem 14, [4].) It is now clear that all D_λ are isomorphic.

Consider the following conditions on a class $\mathcal{C}$ of semigroups:

(A) Every semigroup in $\mathcal{C}$ has an identity.

(B) $\mathcal{C}$ is closed under isomorphisms.

(C) If a semigroup S is an r-composition of semigroups C_λ in $\mathcal{C}$, $\lambda \in \Lambda$, then $R_\Lambda = \{1_\lambda | 1_\lambda \text{ is the identity of } C_\lambda, \lambda \in \Lambda\}$ is a subsemigroup of S (and thus, by the proof of 2.5, $S \cong C_1 \times R_\Lambda$, where C_1 is any of the semigroups C_λ and R_Λ is a right zero semigroup).

Theorem 2.6. *Let $\mathcal{C}$ be a class of semigroups satisfying* (A), (B), (C). *Then every semigroup in $\mathcal{C}$ is right composable.*

Proof. Let $C \in \mathcal{C}$ and suppose that $exe = xe$ for some $e \in C$ and all $x \in C$. Let α be an isomorphism of C onto a semigroup D disjoint from C. In $S = C \cup D$

define multiplication as follows:

$$x*y=\begin{cases} xy & \text{if} \quad x,y\in C \quad \text{or} \quad x,y\in D, \\ [(xe)\alpha]y & \text{if} \quad x\in C, y\in D, \\ (x\alpha^{-1})ey & \text{if} \quad x\in D, y\in C \end{cases}$$

(multiplication in C and D is denoted by juxtaposition). A simple calculation shows that this multiplication is associative. Hence S is an r-composition of C and D. By (B), $D\in\mathscr{C}$ and thus by (C), the identities 1_C and 1_D of C and D, respectively, form a right zero semigroup. Hence

$$e\alpha=[(1_C e)\alpha]1_D=1_C*1_D=1_D,$$

which implies that $e=1_C$. Consequently C is right composable.

Corollary 2. 7. *The class of right composable semigroups is the largest class of semigroups satisfying* (A), (B), (C).

3. The main theorem

Recall that a *rectangular* 0-*band* is a regular Rees matrix semigroup over a one element group, and that a congruence ϱ on a semigroup S is called an I-matrix congruence if S/ϱ is a rectangular 0-band and I is the complete inverse image of 0. The classes of ϱ which are complete inverse images of nonzero idempotents in S/ϱ are called *nonzero classes* of ϱ, the others are *zero classes*. We are interested here solely in the case when S has a zero and $I=0$; in such a case, ϱ is called a 0-*matrix congruence* on S. These concepts were introduced and studied in [3] (see particularly section 1).

Definition 3. 1. Let $\mathscr{C}$ be a class of semigroups. A semigroup S is said to be a 0-*matrix of semigroups of type* $\mathscr{C}$ if $S=S^0$ and there is a 0-matrix congruence $\mathfrak{M}$ on S whose nonzero classes are in $\mathscr{C}$.

Proposition 3. 2. *If* $S=S^0$ *is a semigroup having a* 0-*matrix congruence* $\mathfrak{M}$ *all of whose nonzero classes have an identity, then* $\mathfrak{M}$ *is the finest* 0-*matrix congruence on* S.

Proof. Let $\mathfrak{M}$ be as in the statement of the proposition, and $\Phi(0)$ be the finest 0-matrix congruence on S (2. 6, [3]). If A is a nonzero class of $\mathfrak{M}$, then $\alpha=\Phi(0)|_A$ is a matrix congruence (i.e., A/α is a rectangular band), and since A has an identity, α must be the universal relation. Hence A is a class of $\Phi(0)$. Conversely, if B is a nonzero class of $\Phi(0)$, it must be contained in a nonzero class A of $\mathfrak{M}$ and thus $B=A$, i.e., B is a class of $\mathfrak{M}$. It follows that $\mathfrak{M}$ and $\Phi(0)$ have the same nonzero classes which by 2. 2, [2], implies $\mathfrak{M}=\Phi(0)$.

Definition 3.3. A semigroup S is called a *Rees* 0-*composition* if $S=S^0$ and there is a 0-matrix congruence $\mathfrak{M}$ on S whose classes, denoted by $\Sigma_{i\lambda}$ ($i\in I$, $\lambda\in\Lambda$), satisfy the condition

(D) for every $\mathfrak{M}$-class $\Sigma_{i\lambda}$, there exists an element $x_{i\lambda}\in\Sigma_{i\lambda}$ with the property that for every $j\in I$, $\mu\in\Lambda$:

$$x_{i\lambda}\Sigma_{j\mu}=\Sigma_{i\mu} \text{ or } 0 \text{ and } \Sigma_{j\mu}x_{i\lambda}=\Sigma_{j\lambda} \text{ or } 0.$$

Remarks. i) More precisely, we should speak of a „Rees 0-composition relative to $\mathfrak{M}$"; however, in 3.5 we will prove that every nonzero class of $\mathfrak{M}$ has an identity, which by 3.2 will imply uniqueness of $\mathfrak{M}$.

ii) Note that $\Sigma_{i\lambda}\Sigma_{j\mu}\neq 0$ if and only if $\Sigma_{j\lambda}$ is a nonzero class (p. 80, [3]) so that by (D), $x_{i\lambda}\Sigma_{j\mu}=\Sigma_{i\mu}$ if and only if $\Sigma_{j\lambda}$ is a nonzero class; analogously for $\Sigma_{j\mu}x_{i\lambda}$.

iii) A 0-matrix of semigroups of some type $\mathscr{C}$ need not be a Rees 0-composition; e.g., a 0-matrix of groups is in general an ideal extension of a completely 0-simple semigroup.

Definition 3.4. Let $D=D^1$ be a semigroup with the group of units G (i.e., G is the $\mathscr{H}$-class of 1), and let P be a regular $\Lambda\times I$-matrix over G^0 (i.e., in each row and each column of P there is at least one nonzero entry). By $\mathscr{M}^0(D;I,\Lambda;P)$ denote the set of all elements $(a;i,\lambda)$, with $a\in D^0$ (D with zero adjoined even if D already has a zero), $i\in I$, $\lambda\in\Lambda$ (the elements $(0;i,\lambda)$ are identified with a single element 0, the zero of $\mathscr{M}^0(D;I,\Lambda;P)$) together with the multiplication

$$(a;i,\lambda)(b;j,\mu)=(ap_{\lambda j}b;i,\mu).$$

Then $\mathscr{M}^0(D;I,\Lambda;P)$ is a semigroup which we call the *Rees matrix semigroup* (over D^0). The congruence $\mathfrak{M}$ defined by $(a;i,\lambda)\mathfrak{M}(b;j,\mu)\Leftrightarrow i=j$, $\lambda=\mu$, and $0\mathfrak{M}0$ is called the *associated congruence*.

If D is a group, $D=G$ and our terminology and notation agree with that used in [1] except that we consider only a regular sandwich matrix P. We are now ready to state our main result.

Theorem 3.4. *A semigroup S is a Rees* 0-*composition if and only if S is isomorphic to a Rees matrix semigroup $\mathscr{M}^0(D;I,\Lambda;P)$, where $D=D^1$.*

Proof. Sufficiency. Let $Q=\mathscr{M}^0(D;I,\Lambda;P)$ where $D=D^1$. Note first that the associated congruence $\mathfrak{M}$ on Q is a 0-matrix congruence; its classes different from 0 are the sets $\Sigma_{i\lambda}=\{(a;i,\lambda)\,|\,a\in D\}$, $i\in I$, $\lambda\in\Lambda$. Let $x_{i\lambda}=(1;i,\lambda)$; since $\mathfrak{M}$ is a 0-matrix congruence, $x_{i\lambda}\Sigma_{j\mu}\subseteqq\Sigma_{i\mu}\cup 0$ for any $j\in I$, $\mu\in\Lambda$. If $x_{i\lambda}\Sigma_{j\mu}\neq 0$, then $x_{i\lambda}\Sigma_{j\mu}\subseteqq\Sigma_{i\mu}$ and $p_{\lambda j}\neq 0$. Consequently, for any $(y;i,\mu)\in\Sigma_{i\mu}$, we obtain

$$(y;i,\mu)=(1;i,\lambda)(p_{\lambda j}^{-1}y;j,\mu)\in x_{i\lambda}\Sigma_{j\mu},$$

whence $x_{i\lambda}\Sigma_{j\mu}=\Sigma_{i\mu}$. The other half of condition (D) is established similarly.

Necessity. The proof is broken into several lemmas in which S is a Rees 0-composition, and $\Sigma_{i\lambda}$ are the classes of the congruence induced.

Lemma 3. 5. *Every nonzero class* $\Sigma_{i\lambda}$ *has an identity (denoted by* $1_{i\lambda}$*).*

Proof. Let $\Sigma_{i\lambda}$ be a nonzero class; then $x_{i\lambda}\Sigma_{i\lambda}=\Sigma_{i\lambda}x_{i\lambda}=\Sigma_{i\lambda}$ (by (D)). There is $t\in\Sigma_{i\lambda}$ such that $x_{i\lambda}=x_{i\lambda}t$ and for every $y\in\Sigma_{i\lambda}$, $y=ux_{i\lambda}$ for some $u\in\Sigma_{i\lambda}$. Hence $yt=ux_{i\lambda}t=ux_{i\lambda}=y$, i.e., t is a right identity of $\Sigma_{i\lambda}$. Dually, $\Sigma_{i\lambda}$ also has a left identity which implies that $1_{i\lambda}=t$ is the identity of $\Sigma_{i\lambda}$.

Lemma 3. 6. *If* $y\in\Sigma_{i\lambda}$, *then* $y=1_{i\mu}y=y1_{j\lambda}$ *whenever* $\Sigma_{i\mu}$ *and* $\Sigma_{j\lambda}$ *are nonzero classes.*

Proof. Since $\Sigma_{i\mu}$ is a nonzero class, $x_{i\mu}\Sigma_{i\lambda}=\Sigma_{i\lambda}$ by (D). For any $y\in\Sigma_{i\lambda}$, we obtain $y=x_{i\mu}u$ for some $u\in\Sigma_{i\lambda}$, so that $1_{i\mu}y=1_{i\mu}x_{i\mu}u=x_{i\mu}u=y$. The equality $y=y1_{j\lambda}$ is established analogously.

As a consequence of 3. 6, we have

$$1_{i\mu}1_{i\delta}=1_{i\delta},\ 1_{i\lambda}1_{j\lambda}=1_{i\lambda}$$

provided that $\Sigma_{i\mu}$, $\Sigma_{i\delta}$, $\Sigma_{i\lambda}$, and $\Sigma_{j\lambda}$ are nonzero classes. We will use this without express mention.

Lemma 3. 7. *Let*

$$S_1=\{x\in S|x\mathscr{R}1_{i\nu},x\mathscr{L}1_{k\lambda}\ \textit{for some}\ i,k\in I,\nu,\lambda\in\Lambda\}\cup 0;$$

then S_1 *is a completely* 0*-simple subsemigroup of* S, *and* S_1 *intersects every class of* $\mathfrak{M}$.

Proof. Let $x\in\Sigma_{i\lambda}\cap S_1$ and $y\in\Sigma_{j\mu}\cap S_1$. If $xy=0$, then $xy\in S_1$. Suppose $xy\neq0$. We have $x\mathscr{R}1_{i\nu}$, $x\mathscr{L}1_{k\lambda}$, $y\mathscr{R}1_{j\delta}$, $y\mathscr{L}1_{m\mu}$ for some $k,m\in I$, $\nu,\delta\in\Lambda$. Consequently

$$x1_{j\delta}1_{j\lambda}=x1_{j\lambda}=x1_{k\lambda}1_{j\lambda}=x1_{k\lambda}=x.$$

So we have $x1_{j\delta}\mathscr{R}x$; thus $x\mathscr{R}1_{i\nu}$ implies $x1_{j\delta}\mathscr{R}1_{i\nu}$. Since $\mathscr{R}$ is a left congruence, $y\mathscr{R}1_{j\delta}$ implies $xy\mathscr{R}x1_{j\delta}$, and hence $xy\mathscr{R}1_{i\nu}$. One shows similarly that $xy\mathscr{L}1_{m\mu}$, which proves that $xy\in S_1$. Thus S_1 is a subsemigroup of S.

Let $\Sigma_{i\lambda}$ be any class and $\Sigma_{i\mu}$, $\Sigma_{j\lambda}$ be nonzero classes. Then by (D), $x_{i\lambda}\Sigma_{j\mu}=\Sigma_{i\mu}$ whence $x_{i\lambda}t=1_{i\mu}$ for some $t\in\Sigma_{j\mu}$; this together with $x_{i\lambda}=1_{i\mu}x_{i\lambda}$ (3. 6) implies $x_{i\lambda}\mathscr{R}1_{i\mu}$. Dually, we obtain $x_{i\lambda}\mathscr{L}1_{j\lambda}$, and thus $x_{i\lambda}\in\Sigma_{i\lambda}\cap S_1$, which proves the last statement of the lemma. Further, if $\Sigma_{i\lambda}$ is a nonzero class, then $\Sigma_{i\lambda}\cap S_1=G_{i\lambda}$, the group of units of $\Sigma_{i\lambda}$. For obviously $\Sigma_{i\lambda}\cap S_1\supseteqq G_{i\lambda}$, while the opposite inclusion holds since $x\in\Sigma_{i\lambda}\cap S_1$ implies $x\mathscr{R}1_{i\mu}$, $x\mathscr{L}1_{j\lambda}$ for some $j\in I$, $\mu\in\Lambda$; this together with $1_{i\lambda}\mathscr{R}1_{i\mu}$, $1_{i\lambda}\mathscr{L}1_{j\lambda}$ implies $x\mathscr{H}1_{i\lambda}$. It then follows that the restriction of $\mathfrak{M}$ to

S_1 is a 0-matrix congruence whose nonzero classes are groups. By 3. 6, every element of S_1 has a left (and a right) identity, and thus 4. 1 and 4. 5 of [3] imply that S_1 is completely 0-simple.

Let $H_{i\lambda}=\Sigma_{i\lambda}\cap S_1$ and choose any nonzero class Σ_{11}; then H_{11} is the group of units of Σ_{11}. For each $i\in I, \lambda\in \Lambda$, select $r_i\in H_{i1}$ and $q_\lambda\in H_{1\lambda}$ and define P as the $\Lambda\times I$-matrix $P=(p_{\lambda i})$ over H_{11}^0 by

$$p_{\lambda i}=\begin{cases} q_\lambda r_i & \text{if } q_\lambda r_i\in H_{11},\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 3. 8. *Every nonzero element of S is uniquely representable in the form $r_i a q_\lambda$ with $a\in\Sigma_{11}$, $i\in I$, $\lambda\in\Lambda$ and the mapping Φ defined by $(a; i, \lambda)\Phi=r_i a q_\lambda$, $0\Phi=0$, is an isomorphism of $\mathcal{M}^0(\Sigma_{11}; I, \Lambda; P)$ onto S.*

Proof. For $\lambda\in\Lambda$, there exists $i\in I$ such that $\Sigma_{i\lambda}$ is a nonzero class. Hence q_λ has a unique inverse q'_λ in $R_{1_{i\lambda}}\cap L_{1_{11}}$ since $H_{i\lambda}$ is a group and S_1 is completely 0-simple. Thus $1_{11}q_\lambda=q_\lambda$ and $q_\lambda q'_\lambda=1_{11}$. Now $\mathfrak{M}=\mathfrak{R}\cap\mathfrak{L}$, where $\mathfrak{R}[\mathfrak{L}]$ is a 0-left [0-right] zero equivalence on S (1. 7 and 1. 10, [3]). Let C_i, $i\in I$, and Γ_λ, $\lambda\in\Lambda$, denote the $\mathfrak{R}$ and $\mathfrak{L}$ classes of S, respectively, different from 0. For every $x\in\Gamma_1$, by 3. 6, we obtain $xq_\lambda q'_\lambda=x1_{11}=x$, and analogously, for every $y\in\Gamma_\lambda$, $yq'_\lambda q_\lambda=y$. The mappings $x\to xq_\lambda (x\in\Gamma_1)$ and $y\to yq'_\lambda$ $(y\in\Gamma_\lambda)$ are mutually inverse C_i-class preserving one-one mappings of Γ_1 onto Γ_λ and of Γ_λ onto Γ_1, respectively. Using r_i and r'_i, one similarly establishes one-one Γ_λ-class preserving correspondences between C_1 and C_i. It follows that the mappings $x\to r_i x q_\lambda$ $(x\in\Sigma_{11})$ and $y\to r'_i y q'_\lambda$ $(y\in\Sigma_{i\lambda})$ are one-one inverse mappings. Since every nonzero element of S belongs to some $\Sigma_{i\lambda}$, this proves the first part of the lemma and also that Φ is one-one and onto. The proof that Φ is a homomorphism is the same as for the corresponding part of the Rees theorem in [1], pages 93 and 94.

This completes the proof of 3. 4.

Recall that a *matrix congruence* ϱ on a semigroup S is a congruence such that S/ϱ is a rectangular band (see, e.g., [4]). If we adjoin a zero to S and extend ϱ to S^0 by letting $0\varrho 0$, we get a 0-matrix congruence. Definitions 3. 1 and 3. 3 then carry over to this case if we then remove the zero. We thus obtain a *matrix of semigroups of type $\mathscr{C}$* and a *Rees composition $\mathcal{M}(D; I, \Lambda; P)$*. The next theorem shows that for the case of a matrix of semigroups, the class of composable semigroups is the best in a certain sense.

Theorem 3. 9. *Let $\mathscr{C}$ be a class of semigroups closed under isomorphisms. Then every semigroup in $\mathscr{C}$ has an identity and every matrix of semigroups of type $\mathscr{C}$ is a Rees composition if and only if $\mathscr{C}$ is contained in the class of composable semigroups.*

Proof. Necessity. Let S be an r-composition of semigroups C_λ in $\mathscr{C}$, $\lambda \in \Lambda'$. By hypothesis and 3.5, $S \cong \mathscr{M}(D; I, \Lambda; P)$ with $D = D^1$. Since every C_λ and D have identities, 3.2 implies that, by identifying S with $\mathscr{M}(D; I, \Lambda; P)$, the congruences induced by the r-composition and by the Rees composition coincide. Hence we may set $I = \{1\}$, $\Lambda = \Lambda'$. If 1_λ is the identity of C_λ, we have $1_\lambda = (p_{\lambda 1}^{-1}; 1, \lambda)$; it follows that the set $R_\Lambda = \{1_\lambda | \lambda \in \Lambda\}$ is a subsemigroup of S. We have proved that $\mathscr{C}$ satisfies condition (C) (preceding 2.6); since $\mathscr{C}$ satisfies (A) and (B) by hypothesis, 2.6 implies that every semigroup in $\mathscr{C}$ is right composable. A dual proof shows that every semigroup in $\mathscr{C}$ is also left composable.

Sufficiency. Let S be a matrix of composable semigroups $\Sigma_{i\lambda}$ with identity $1_{i\lambda}$, $i \in I$, $\lambda \in \Lambda$. To establish condition (D) in this case, it suffices to show that $1_{i\lambda}\Sigma_{j\mu} = \Sigma_{i\mu}$ and $\Sigma_{j\mu}1_{i\lambda} = \Sigma_{j\lambda}$ for all $i, j \in I$, $\lambda, \mu \in \Lambda$. The set $C_i = \bigcup_{\lambda \in \Lambda} \Sigma_{i\lambda}$ is an r-composition of semigroups $\Sigma_{i\lambda}$ which are (right) composable; 2.5 then implies $1_{i\lambda}1_{i\mu} = 1_{i\mu}$; dually, we have $1_{i\lambda}1_{j\lambda} = 1_{i\lambda}$. Hence

$$1_{i\mu} = 1_{i\lambda}1_{i\mu} = (1_{i\lambda}1_{j\lambda})1_{i\mu} = 1_{i\lambda}(1_{j\lambda}1_{i\mu}) \in 1_{i\lambda}\Sigma_{j\mu},$$

whence for all $x \in \Sigma_{i\mu}$,

$$x = 1_{i\mu}x \in 1_{i\lambda}\Sigma_{j\mu}x \subseteq 1_{i\lambda}\Sigma_{j\mu}.$$

Consequently $\Sigma_{i\mu} \subseteq 1_{i\lambda}\Sigma_{j\mu}$; the opposite inclusion holds since $\Sigma_{i\lambda}\Sigma_{j\mu} \subseteq \Sigma_{i\mu}$. Thus $1_{i\lambda}\Sigma_{j\mu} = \Sigma_{i\lambda}$; the equality $\Sigma_{j\mu}1_{i\lambda} = \Sigma_{j\lambda}$ is proved symmetrically. Therefore S is a Rees composition.

Corollary 3.10. *A semigroup S is a matrix of composable semigroups if and only if $S \cong \mathscr{M}(D; I, \Lambda; P)$, where D is composable.*

It appears to be much more difficult to obtain a characterization of a 0-matrix of semigroups of type $\mathscr{T}$ without additional restrictions. The next theorem, which generalizes 4.5, [3], points in this direction.

Theorem 3.11. *Let S be a 0-matrix of bisimple semigroups with identity. Then the following conditions on S are equivalent:*

a) *S is regular.*

b) *S is 0-bisimple.*

c) *S is a Rees 0-composition.*

In such a case, $S \cong \mathscr{M}^0(D; I, \Lambda; P)$, where $D = D^1$ is bisimple.

Proof. Denote the classes of the 0-matrix congruence (see 3.2) by $\Sigma_{i\lambda}$, $i \in I$, $\lambda \in \Lambda$, and if $\Sigma_{i\lambda}$ is a nonzero class, let $1_{i\lambda}$ denote its identity. Recall the notation $C_i = \bigcup_{\lambda \in \Lambda} \Sigma_{i\lambda}$, $\Gamma_\lambda = \bigcup_{i \in I} \Sigma_{i\lambda}$.

a)$\Rightarrow$b). If $x \in \Sigma_{i\lambda}$, then by regularity of S, $x = xyx$ for some $y \in \Sigma_{j\mu}$. It follows that $e = yx$ is an idempotent of $\Sigma_{j\lambda}$ and $x\mathscr{L}e$. Since $\Sigma_{j\lambda}$ is then a bisimple semigroup, we have $e\mathscr{D}1_{j\lambda}$, and thus $x\mathscr{D}1_{j\lambda}$. If $\Sigma_{k\lambda}$ is any nonzero class, 2. 3 and 2. 5 imply $1_{j\lambda}1_{k\lambda}=1_{j\lambda}$ and $1_{k\lambda}1_{j\lambda}=1_{k\lambda}$, i.e., $1_{j\lambda}\mathscr{R}1_{k\lambda}$. Thus $x\mathscr{D}1_{k\lambda}$ which shows that any two elements of Γ_λ are $\mathscr{D}$-equivalent. By symmetry we obtain that any two elements of C_i are also $\mathscr{D}$-equivalent. Since these statements hold for any i, λ it follows that S is 0-bisimple.

b)⇒c). Consider any $\Sigma_{i\lambda}$ and any nonzero classes $\Sigma_{i\nu}$ and $\Sigma_{k\lambda}$. Since S is 0-bisimple, there exists $x\in S$ such that $1_{i\nu}\mathscr{R}x$ and $x\mathscr{L}1_{k\lambda}$. It follows that $x\in\Sigma_{i\lambda}\cap S_1$. Let $x_{i\lambda}$ be any element of $\Sigma_{i\lambda}\cap S_1$ and suppose that $x_{i\lambda}\Sigma_{j\mu}\neq 0$; then $x_{i\lambda}\Sigma_{j\mu}\subseteq\Sigma_{i\mu}$. Let $y\in\Sigma_{i\mu}$. Since S is 0-bisimple and contains nonzero idempotents, S is regular and thus $y=ey$ for some idempotent $e\in\Sigma_{i\theta}$. Hence $\Sigma_{i\theta}$ is a nonzero class and thus $x_{i\lambda}\mathscr{R}1_{i\theta}$, which implies $1_{i\theta}=x_{i\lambda}z$ for some z, and $x_{i\lambda}=1_{i\theta}x_{i\lambda}$. By symmetry, we have $x_{i\lambda}=x_{i\lambda}1_{n\lambda}$ for some $n\in I$, which together with $1_{j\lambda}\mathscr{L}1_{n\lambda}$ implies $x_{i\lambda}=x_{i\lambda}1_{j\lambda}$. Consequently

$$y = ey = 1_{i\theta}(ey) = 1_{i\theta}y = (x_{i\lambda}z)y = (x_{i\lambda}1_{j\lambda})zy = x_{i\lambda}(1_{j\lambda}zy)\in x_{i\lambda}\Sigma_{j\mu}.$$

Therefore $\Sigma_{i\mu}\subseteq x_{i\lambda}\Sigma_{j\mu}$ and the equality holds. The proof of $\Sigma_{j\mu}x_{i\lambda}=\Sigma_{j\lambda}$, if $\Sigma_{i\mu}$ is a nonzero class, is dual. Therefore (D) holds and S is a Rees 0-composition. ∎

c)⇒a). By 3. 4, $S\cong\mathscr{M}^0(D; I, \Lambda; P)$ with $D=D^1$, and by the uniqueness of induced congruences (3. 2), D is bisimple. Item a) then follows by a straightforward computation in $\mathscr{M}^0(D; I, \Lambda; P)$ using regularity of D.

4. Homomorphisms of Rees matrix semigroups

A homomorphism φ of a semigroup $S=S^0$ into a semigroup $T=T^0$ is said to be *0-restricted* if $a\varphi=0\Leftrightarrow a=0$. A homomorphic image of a Rees matrix semigroup need not be a Rees matrix semigroup; however, if φ is a 0-restricted homomorphism of a Rees matrix semigroup S onto S^*, then S^* is also a Rees matrix semigroup. The next theorem describes all 0-restricted homomorphisms of a Rees matrix semigroup into another; it generalizes a result of MUNN (3. 11, [1]). Recall that for a semigroup D, D^0 denotes the semigroup obtained by adjoining a zero to D (even if D already has a zero).

Theorem 4. 1. *Let $S=\mathscr{M}^0(D; I, \Lambda; P)$, $S^*=\mathscr{M}^0(D^*; I^*, \Lambda^*; P^*)$, where D and D^* are semigroups with identities 1 and 1^*, respectively. Let ω be a 0-restricted homomorphism of D^0 into $(D^*)^0$. Let $i\to u_i$ be a mapping of I into the $\mathscr{R}$-class of 1ω, $\lambda\to v_\lambda$ be a mapping of Λ into the $\mathscr{L}$-class of 1ω, and Φ, ψ be mappings of I into I^* and Λ into Λ^*, respectively, such that*

$$p_{\lambda i}\omega = v_\lambda p^*_{\lambda\psi, i\Phi}u_i \tag{1}$$

for all $i \in I$, $\lambda \in \Lambda$. *For each element* $(a; i, \lambda) \in S$, *define*

(2) $$(a; i, \lambda)\theta = [u_i(a\omega)v_\lambda; i\Phi, \lambda\psi].$$

Then θ *is a* 0-*restricted homomorphism of* S *into* S^*. *Conversely, every* 0-*restricted homomorphism of* S *into* S^* *can be obtained in this fashion.*

Proof. In the direct part, the proof that θ is a homomorphism is the same as in 3. 11, [1], and is omitted. It is clear that θ is 0-restricted.

For the converse, the proof of 3. 11, [1], is modified as follows. The mappings Φ and ψ are defined as there (substituting $\mathscr{R}$ and $\mathscr{L}$-classes by $\mathfrak{R}$ and $\mathfrak{L}$-classes, respectively; see the proof of 3. 8). We select a nonzero class Σ_{11} of the associated congruence $\mathfrak{M}$ of S, and denote its identity by 1_{11}. Then $1_{11}\theta$ is a nonzero idempotent so that the class of $\mathfrak{M}^*$ (the associated congruence of S^*) is nonzero, whence $p^*_{1\psi, 1\Phi} \neq 0$. The equation

(3) $$(p_{11}^{-1}x; 1, 1)\theta = [p^{*-1}_{1\psi, 1\Phi}(x\omega); 1\Phi, 1\psi]$$

defines a homomorphism of D into D^*. For every $i \in I$, define u_i by

(4) $$(1; i, 1)\theta = [u_i; i\Phi, 1\psi]$$

and for every $\lambda \in \Lambda$, define v_λ by

(5) $$(p_{11}^{-1}; 1, \lambda)\theta = [p^{*-1}_{1\psi, 1\Phi}v_\lambda; 1\Phi, \lambda\psi].$$

Since $(1; i, 1)\mathscr{L}(p_{11}^{-1}; 1, 1)$, by (3) and (4), we obtain

$$[u_i; i\Phi, 1\psi]\mathscr{L}[p^{*-1}_{1\psi, 1\Phi}(1\omega); 1\Phi, 1\psi],$$

which implies $u_i\mathscr{L}1\omega$. Similarly $(p_{11}^{-1}; 1, 1)\mathscr{R}(p_{11}^{-1}; 1, \lambda)$ implies, by (3) and (5),

$$[p^{*-1}_{1\psi, 1\Phi}v_\lambda; 1\Phi, \lambda\psi]\mathscr{R}[p^{*-1}_{1\psi, 1\Phi}(1\omega); 1\Phi, 1\psi],$$

which implies $p^{*-1}_{1\psi, 1\Phi}v_\lambda\mathscr{R}p^{*-1}_{1\psi, 1\Phi}(1\omega)$ whence $v_\lambda\mathscr{R}1\omega$. Writing $(a; i, \lambda) \in S$ in the form

$$(1; i, 1)(p_{11}^{-1}a; 1, 1)(p_{11}^{-1}; 1, \lambda)$$

and applying θ, we obtain (2). From (2), we have

$$(1; i, \lambda)^2\theta = [u_i(p_{\lambda i}\omega)v_\lambda; i\Phi, \lambda\psi],$$

$$[(1; i, \lambda)\theta]^2 = [u_i(1\omega)v_\lambda p^*_{\lambda\psi, i\Phi}u_i(1\omega)v_\lambda; 1\Phi, \lambda\psi]$$

and thus

(6) $$u_i(p_{\lambda i}\omega)v_\lambda = u_i(1\omega)v_\lambda p^*_{\lambda\psi, i\Phi}u_i(1\omega)v_\lambda.$$

Since $u_i\mathscr{L}1\omega$ and $v_\lambda\mathscr{R}1\omega$, we have $u_i(1\omega) = u_i$, $u_i'u_i = 1\omega$, $(1\omega)v_\lambda = v_\lambda$, $v_\lambda v_\lambda' = 1\omega$ for some $u_i', v_\lambda' \in D$. Taking into account $u_i(1\omega) = u_i$, $(1\omega)v_\lambda = v_\lambda$ and multiplying (6) on the left by u_i' and on the right by v_λ', we obtain (1).

To state the next corollary, using the notation of 4. 1, we define a left invertible

$I^* \times I$-matrix U over $(D^*)^0$ as a matrix which has exactly one nonzero entry in each row and in each column, this entry being in the $\mathcal{L}$-class of 1^*. A right invertible $\Lambda \times \Lambda^*$-matrix V is defined dually. The proof of the following corollary is essentially the same as the proof of 3. 12, [1].

Corollary 4. 2. *Two Rees matrix semigroups $\mathcal{M}^0(D; I, \Lambda; P)$ and $\mathcal{M}^0(D^*; I^*, \Lambda^*; P^*)$ are isomorphic if and only if there exists an isomorphism ω of D^0 onto $(D^*)^0$, a left invertible $I^* \times I$-matrix U over $(D^*)^0$ and a right invertible $\Lambda \times \Lambda^*$-matrix V over $(D^*)^0$ such that $P\omega = VP^*U$.*

We now consider the special cases of Rees matrix semigroups which can be conveniently expressed as products of certain semigroups. Let A and B be semigroups, where B has a zero 0. By $A \times^0 B$ denote the Rees quotient $A \times B / A \times 0$ ($A \times B$ is the Cartesian product of A and B). Let P be a $\Lambda \times I$-matrix over a group with zero G^0. We say that P satisfies condition (M) if every nonzero product of the form

$$p_{\lambda_1 i_1}^{-1} p_{\lambda_1 i_2} p_{\lambda_2 i_2}^{-1} p_{\lambda_2 i_3} \cdots p_{\lambda_{n-1} i_{n-1}}^{-1} p_{\lambda_{n-1} i_n} p_{\lambda_n i_n}^{-1} p_{\lambda_n i_1}$$

is equal to 1, the identity of G (p. 97, [3]). Recall the definition of S_1 (3. 7).

Theorem 4. 3. *Let $S = \mathcal{M}^0(D; I, \Lambda; P)$ and let G be the group of units of $D = D^1$. Let $\bar{P}$ be the $\Lambda \times I$-matrix with entries*

$$\bar{p}_{\lambda i} = \begin{cases} 1 & \text{if } p_{\lambda i} \neq 0 \\ 0 & \text{if } p_{\lambda i} = 0. \end{cases}$$

Let $B = \mathcal{M}^0(1; I, \Lambda; \bar{P})$, where 1 denotes a one element group. Then $S_1 = \mathcal{M}^0(G; I, \Lambda; P)$ and the following statements are equivalent:

a) $S \cong D \times^0 B$; b) $S_1 \cong G \times^0 B$; c) *P satisfies (M).*

Proof. The first statement follows easily from the proof of 3. 7; b) and c) are equivalent by 4. 13, [3] ((a)$\Leftrightarrow$(e)). Since $S_1 = \mathcal{M}^0(G; I, \Lambda; P)$, it follows easily that a) implies b). Suppose that c) holds. By 4. 13 and 4.10 of [3], there exists a subsemigroup F of S_1 intersecting every $\mathcal{H}$-class $H_{i\lambda}$ of S_1 in exactly one element; denote it by $e_{i\lambda}$. If $H_{i\lambda}$ is a group, $e_{i\lambda}$ is an idempotent and thus $e_{i\lambda} = (p_{\lambda i}^{-1}; i, \lambda)$. If for $(x; i, \lambda) \in S$, $(x; i, \lambda)e_{j\lambda} \neq 0$, then $p_{\lambda j} \neq 0$ and thus $e_{j\lambda} = (p_{\lambda j}^{-1}; j, \lambda)$. Consequently $(x; i, \lambda)e_{j\lambda} = (x; i, \lambda)$. Symmetrically, if $e_{i\mu}(x; i, \lambda) \neq 0$, then $e_{i\mu}(x; i, \lambda) = (x; i, \lambda)$. Applying 4. 8, [3], we obtain a).

Corollary 4. 4. *Let $S = \mathcal{M}^0(D; I, \Lambda; P)$. If the group of units of D is trivial, then $S \cong D \times^0 B$, where B is as in 4. 3.*

Proof. If the group of units of D is trivial, 4. 3 implies that $S_1 \cong B$, whence $S \cong D \times^0 B$ again by 4. 3.

5. Rees matrix semigroups over a bisimple inverse semigroup with identity

The principal object of this section is to give an abstract characterization of such a semigroup using certain properties of its set of idempotents. From this we then derive simple characterizations of several classes of semigroups. The set E_S of idempotents of a semigroup S is now considered as a partially ordered set under the usual order $e \leqq f \Leftrightarrow e = ef = fe$. If we write $E_S \cong C$, where C is a semigroup, it means that E_S is a subsemigroup of S and is isomorphic to C.

Theorem 5.1. *Let S be a 0-bisimple semigroup. Then $S \cong \mathcal{M}^0(D; I, \Lambda; P)$, where $D = D^1$ is a bisimple inverse semigroup if and only if S satisfies:*

a) *for all $a, b, c \in S$, $abc = 0 \Rightarrow ab = 0$ or $bc = 0$;*
b) *there exist order isomorphisms φ and ψ of E_S onto E_A,*

where $A = T \times {}^0 B$, $T = T^1$ is a semilattice, B is a rectangular 0-band, such that for all $e, f \in E_S$,

i) *$ef = f \Leftrightarrow (e\varphi)(f\varphi) = f\varphi$,*
ii) *$ef = e \Leftrightarrow (e\psi)(f\psi) = e\psi$,*
iii) *if $e\varphi = (x, a)$ and $e\psi = (y, b)$, then $a = b$;*
c) *for all $e, f \in E_S$,*
i) *$eS \cap fS \neq 0 \Rightarrow eS \cap fS = efS$,*
ii) *$Se \cap Sf \neq 0 \Rightarrow Se \cap Sf = Sef$.*
In such a case, $T \cong E_D$, $B \cong \mathcal{M}^0(1; I, \Lambda; \bar{P})$, where $\bar{P}$ is as in 4.3.

Proof. Necessity. For convenience we identify S with $\mathcal{M}^0(D; I, \Lambda; P)$. Item a) follows from the fact that the associated congruence $\mathfrak{M}$ is a 0-matrix congruence (1.6, [3]). Let $T = E_D$, $B = \mathcal{M}^0(1; I, \Lambda; \bar{P})$, and $A = T \times {}^0 B$. It is easy to see that

(1) $$E_S = \{(x; i, \lambda) \mid p_{\lambda i} \neq 0,\ x p_{\lambda i} x = x\} \cup 0.$$

On E_S define the mappings φ and ψ by:

$$(x; i, \lambda)\varphi = \big(x p_{\lambda i}, (1; i, \lambda)\big) \text{ if } x \neq 0, \text{ and } 0\varphi = 0,$$

$$(x; i, \lambda)\psi = \big(p_{\lambda i} x, (1; i, \lambda)\big) \text{ if } x \neq 0, \text{ and } 0\psi = 0.$$

Note that

(2) $$E_A = \{(e, (1; i, \lambda)) \mid e \in T, p_{\lambda i} \neq 0\} \cup 0.$$

Using (1) and (2), it is straightforward to verify that φ and ψ satisfy all the conditions in b). We prove only that c) i) holds; c) ii) is treated analogously. Thus let $e = (x; i, \lambda)$, $f = (y; j, \mu)$ be idempotents of S such that $eS \cap fS \neq 0$. Then

$$(x; i, \lambda)(z; k, \nu) = (y; j, \mu)(w; m, \delta) \neq 0$$

for some $(z; k, v)$, $(w; m, \delta) \in S$ and hence $i=j$. Since $e, f \in E_S$, (1) yields $p_{\lambda i} \neq 0$, $p_{\mu i} = p_{\mu j} \neq 0$, which by commutativity of idempotents in D implies

$$x p_{\lambda i} y = (x p_{\lambda i})(y p_{\mu i}) p_{\mu i}^{-1} = (y p_{\mu i})(x p_{\lambda i}) p_{\mu i}^{-1}.$$

Consequently

$$ef = (x; i, \lambda)(y; i, \mu) = (x p_{\lambda i} y; i, \mu) = (y; i, \mu)(x p_{\lambda i} p_{\mu i}^{-1}; i, \mu)$$

which implies $ef \in eS \cap fS$. Conversely, if $x \in eS \cap fS$, then $x = ex = fx = efx \in efS$. Therefore $eS \cap fS = efS$.

Sufficiency. We will freely use the terminology and results of [3]. First note that S is regular (0-bisimple containing a nonzero idempotent). Since S is 0-bisimple, for $a \neq 0$, $b \neq 0$, there is $c \in S$ such that $aS = cS$, $Sc = Sb$. It follows $aSb = cSb = cSc \neq 0$ and 0 is a prime ideal of S, which together with a) implies that 0 is a matrix ideal of S (p. 74, [3]). By 1. 6, [3], S has a 0-matrix congruence; let $\mathfrak{M}$ be the finest such. Then $\mathfrak{M} = \sigma \cap \tau$, where $a \sigma b \Leftrightarrow a = b = 0$ or there exist $a_1, a_2, \cdots, a_n \in S$ such that

$$aS \setminus 0 \,|\, a_1 S \setminus 0 \,|\, \cdots \,|\, a_n S \setminus 0 \,|\, bS \setminus 0 \tag{3}$$

($|$ means "intersects") and τ is defined symmetrically using left ideals (2. 6, [3]). We will show that each nonzero class of $\mathfrak{M}$ is a bisimple inverse semigroup with identity.

If $B \cong \mathcal{M}^0(1; I, \Lambda; Q)$, then $A \cong \mathcal{M}^0(T; I, \Lambda; Q)$. We identify A with $\mathcal{M}^0(T; I, \Lambda; Q)$.

Suppose that for $e, f \in E_S$, $eS \cap fS \neq 0$. By c) i), $eS \cap fS = efS$ which implies $f(ef) = ef = e(ef)$. Thus $ef \in E_S$ so that by b) i), we obtain

$$(f\varphi)[(ef)\varphi] = (ef)\varphi = (e\varphi)[(ef)\varphi] \neq 0.$$

Hence if $e\varphi = (t; i, \lambda)$ and $f\varphi = (u; j, \mu)$, then $i = j$. Now, if $e, f \in E_S$, $e \neq 0$, $e \sigma f$, then by (3)

$$eS \setminus 0 \,|\, e_1 S \setminus 0 \,|\, \cdots \,|\, e_n S \setminus 0 \,|\, fS \setminus 0$$

for some $e_1, e_2, \cdots, e_n \in E_S$ since $a_i S = e_i S$ for some $e_i \in E_S$ by regularity of S. Letting again $e\varphi = (t; i, \lambda)$ and $f\varphi = (u; j, \mu)$, the preceding observation implies $i = j$. Dually, if $e \tau f$, $e\psi = (v; i, \lambda)$, $f\psi = (w; j, \mu)$, then $\lambda = \mu$.

Conversely, if $e\varphi = (t; i, \lambda)$ and $f\varphi = (u; i, \mu)$, then $(e\varphi)[(e\varphi)(f\varphi)] = (e\varphi)(f\varphi)$ and $(f\varphi)[(e\varphi)(f\varphi)] = (e\varphi)(f\varphi)$, which by b) i) implies

$$e[(e\varphi)(f\varphi)]\varphi^{-1} = [(e\varphi)(f\varphi)]\varphi^{-1} = f(e\varphi)(f\varphi)]\varphi^{-1},$$

i.e., $eS \cap fS \neq 0$. Dually $e\psi = (v; i, \lambda)$, $f\psi = (w; j, \lambda)$ implies $Se \cap Sf \neq 0$. Consequently

$$e \sigma f \Leftrightarrow e\varphi = (t; i, \lambda) \text{ and } f\varphi = (u; i, \mu), \tag{4}$$

$$e \tau f \Leftrightarrow e\psi = (v; i, \lambda) \text{ and } f\psi = (w; j, \lambda). \tag{5}$$

Using (4) and (5) we now show that the classes of $\mathfrak{M}$ different from 0 can be indexed by the set $\Lambda \times I$. Let $C \neq 0$ be a σ-class; then C contains an $\mathcal{R}$-class, and since S is regular, C also contains an idempotent e. If $e\varphi = (t; i, \lambda)$, write $C = C_i$. By (4), the index i is independent of the choice of the idempotent in C. If $C_i = C_j$, then clearly $i = j$. Further, for any $i \in I$, there is an idempotent $(t; i, \lambda) \in A$ for some $\lambda \in \Lambda$. Since φ is onto, there is $e \in E_S$ such that $e\varphi = (t; i, \lambda)$. But then the σ-class containing e has index i. We have proved that I can be used as an index set for the σ-classes distinct from 0. Similarly, the set of τ-classes Γ distinct from 0 can be indexed by Λ. Consequently, the $\mathfrak{M}$-classes distinct from 0 can be written as $\Sigma_{i\lambda} = C_i \cap \Gamma_\lambda$ with $i \in I$, $\lambda \in \Lambda$.

If $\Sigma_{i\lambda}$ is a nonzero $\mathfrak{M}$-class and $a \in \Sigma_{i\lambda}$, then $a^2 \in \Sigma_{i\lambda}$. For b an inverse of a^2, we obtain $e = aba \in E_S$, whence $e\varphi = (t; i, \lambda)$ is an idempotent of A and $q_{\lambda i} \neq 0$. Conversely, if $q_{\lambda i} \neq 0$, for any $t \in T$, $(t; i, \lambda) \in E_A$ and thus $(t; i, \lambda)\varphi^{-1} = e \in C_i$. Hence $e\varphi = (t; i, \lambda)$ and by b) iii), $e\psi = (u; i, \lambda)$ so that $e \in \Gamma_\lambda$. Thus $e \in \Sigma_{i\lambda}$ which is then a nonzero $\mathfrak{M}$-class.

For the remainder of the proof let $\Sigma_{i\lambda}$ be a nonzero $\mathfrak{M}$-class. Let $a \in \Sigma_{i\lambda}$; then a has an inverse $a' \in \Sigma_{j\mu}$ for some $j \in I$, $\mu \in \Lambda$, and aa', $a'a$ are idempotents. Since $aa' \in \Sigma_{i\mu}$ and $a'a \in \Sigma_{j\lambda}$, we have $(aa')\varphi = (m; i, \mu)$ and $(a'a)\varphi = (n; j, \lambda)$ for some $m, n \in T$. In A, $(n; i, \lambda)$ and $(m; i, \lambda)$ are nonzero idempotents. Letting $f = (n; i, \lambda)\psi^{-1}$ and $g = (m; i, \lambda)\varphi^{-1}$, we obtain

$$[(a'a)\psi](f\psi) = (n; j, \lambda)(n; i, \lambda) = (n; j, \lambda) = (a'a)\psi,$$

which by b) ii) implies $(a'a)f = a'a$, so that $af = a$. Analogously, using b) i), we derive $ga = a$. Thus $a = aa'a = (af)a'(ga) = a(fa'g)a$, where $fa'g \in \Sigma_{i\lambda}$. Therefore $\Sigma_{i\lambda}$ is regular.

If $a, b \in \Sigma_{i\lambda}$, there exists $c \in S$ such that $aS = cS$, $Sb = Sc$, S being 0-bisimple. Clearly $c \in \Sigma_{i\lambda}$. Letting a', b', c' be any inverses in $\Sigma_{i\lambda}$ of a, b, c, respectively, we obtain $a = cc'a$, $c = aa'c$ which proves $a\Sigma_{i\lambda} = c\Sigma_{i\lambda}$, and $c = cb'b$, $b = bc'c$ which proves $\Sigma_{i\lambda}c = \Sigma_{i\lambda}b$. Hence $\Sigma_{i\lambda}$ is bisimple.

We show next that the idempotents of $\Sigma_{i\lambda}$ commute. Thus let $e, f \in E_S \cap \Sigma_{i\lambda}$. Since, $e, f \in \Sigma_{i\lambda}$, we have $e\sigma f$, $e\tau f$, which by (4) and (5) yields $e\varphi = (t; i, \lambda)$, $f\varphi = (u; i, \mu)$, $e\psi = (v; j, \nu)$, $f\psi = (w; k, \nu)$ for some $t, u, v, w \in T$, $i, j, k \in I$, $\lambda; \mu, \nu \in \Lambda$. By b) iii), $i = j = k$, $\lambda = \mu = \nu$. Thus $e\varphi$, $f\varphi$, $e\psi$, $f\psi$ commute. Let $z \in \Sigma_{i\lambda}$ be an inverse of ef; for $g = fze$, we have $g \in E_S \cap \Sigma_{i\lambda}$ and $ge = g$. It follows that $eg \in E_S$ and $g(eg) = g$, which by b) ii) implies $(g\psi)[(eg)\psi] = g\psi$. Similarly $(eg)g = eg$ implies $[(eg)\psi](g\psi) = (eg)\psi$. Since eg, $g \in \Sigma_{i\lambda}$, $(eg)\psi$ and $g\psi$ commute and thus $(eg)\psi = g\psi$. Consequently $eg = g$, i.e., $efze = fze$. Hence $ef = efzef = fzef$ so that $fef = ef$ and $ef \in E_S$. By symmetry, we conclude that $efe = fe$, whence $fe \in E_S$. Further, $fef = (fe)(ef) = ef$ implies $[(fe)\varphi][(ef)\varphi] = (ef)\varphi$ and $efe = (ef)(fe) = fe$ implies $[(ef)\varphi][(fe)\varphi] = (fe)\varphi$ by b) i). Since $(fe)\varphi$ and $(ef)\varphi$ commute, we obtain $(ef)\varphi = (fe)\varphi$, whence $ef = fe$

From the above, we also see that $(1; i, \lambda)\varphi^{-1}$ is a left identity of $E_S \cap \Sigma_{i\lambda}$. Hence for $a \in \Sigma_{i\lambda}$ and its inverse $a^{-1} \in \Sigma_{i\lambda}$ (unique), we obtain

$$(1; i, \lambda)\varphi^{-1}a = [(1; i, \lambda)\varphi^{-1}]aa^{-1}a = a'a = a.$$

Analogously $(1; i, \lambda)\psi^{-1}$ is a right identity of $\Sigma_{i\lambda}$. Therefore $\Sigma_{i\lambda}$ has an identity.

We have proved that every nonzero $\mathfrak{M}$-class is a bisimple inverse semigroup with identity. By 3. 11, S is a Rees 0-composition, and since every nonzero $\mathfrak{M}$-class has an identity, by 3. 2, $\mathfrak{M}$ must be the congruence associated to S. Therefore $S \cong \mathcal{M}^0(D; I, \Lambda; P)$, where $D = D^1$ is a bisimple inverse semigroup. For $q_{\lambda i} \neq 0$, $\varphi | E_S \cap \Sigma_{i\lambda}$ is a semigroup isomorphism of $E_S \cap \Sigma_{i\lambda}$ onto $T_{i\lambda} = \{(t; i, \lambda) | t \in T\}$. Thus

$$E_D \cong E_S \cap \Sigma_{i\lambda} \cong T_{i\lambda} \cong T.$$

It was shown above that $q_{i\lambda} \neq 0 \Leftrightarrow \Sigma_{i\lambda}$ is a nonzero $\mathfrak{M}$-class. Hence $Q = \bar{P}$. This completes the proof.

Remark 5. 2. In the last part of the proof of necessity, we have in fact shown that $eS \cap fS \subseteqq efS$ always holds. A simple computation then shows that $eS \cap fS = efS$ in c) i) can be substituted by any one of the following expressions: (a) $efS \subseteqq fS$, (b) $ef \in fS$, (c) $ef = fef$.

In order to express conveniently the corollaries of 5. 1, we now introduce the notion of a sum of rectangular bands.

Proposition 5. 3. *Let* $C = C^0$ *be a semigroup,* $|C| > 1$; *then the following conditions are equivalent:*

a) *C is a band and for all $a, b, c \in C$, $ab \neq 0$, $bc \neq 0 \Rightarrow abc = ac \neq 0$;*

b) *C is an orthogonal sum of semigroups B_α^0, where B_α are pairwise disjoint rectangular bands* (5. 11, [3]; *i.e., C is the union of its subsemigroups B_α and* 0, *and* $B_\alpha B_\beta = 0$ *if* $\alpha \neq \beta$);

c) *$C \cong E_B$ for some rectangular* 0*-band* B.

Such a semigroup C will be called a *sum of rectangular bands.*

Proof. a)$\Rightarrow$b). For $a \neq 0$, $b \neq 0$, let: $a\tau b \Leftrightarrow ab \neq 0$. Then τ is an equivalence relation whose classes B_α are rectangular bands, and C is evidently an orthogonal sum of B_α^0.

b)$\Rightarrow$c). We may put $B_\alpha = \mathcal{M}(1; I_\alpha, \Lambda_\alpha; P_\alpha)$, where the sets I_α (respectively Λ_α) are pairwise disjoint. Let $I = \bigcup_\alpha I_\alpha$, $\Lambda = \bigcup_\alpha \Lambda_\alpha$; let $Q = (q_{\lambda i})$ be the $\Lambda \times I$-matrix defined by: for $i \in I_\alpha$, $\lambda \in \Lambda_\beta$

$$q_{\lambda i} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta, \end{cases}$$

and set $B = \mathcal{M}^0(1; I, \Lambda; Q)$. Clearly B is a rectangular 0-band and $C \cong E_B$.

c)$\Rightarrow$a). Let B be a rectangular 0-band and suppose that E_B is a subsemigroup of B. Letting $C=E_B$ and using the hypothesis that E_B is a semigroup, we see without difficulty that C satisfies a).

Definition 5.4. Let P be a $\Lambda\times I$-matrix over a group with zero G^0 and identity 1. P is said to satisfy condition (N) if for all $i,j\in I$, $\lambda,\mu\in\Lambda$,

$$p_{\lambda i}\neq 0, \quad p_{\lambda j}\neq 0, \quad p_{\mu j}\neq 0 \Rightarrow p_{\lambda i}^{-1}p_{\lambda j}p_{\mu j}^{-1}p_{\mu i} = 1.$$

The statement of 5.1 simplifies considerably if we suppose that the idempotents of S form a subsemigroup, or that S has no zero, or that the associated congruence is a Brandt congruence (3.2, [3]). Also, using 4.3, we obtain certain other characterizations of these semigroups; 5.1 thus has the following corollaries.

Corollary 5.5. *The following statements are equivalent for any semigroup S:*

a) *S is 0-bisimple and $E_S\cong T\times^0 C$, where $T=T^1$ is a semilattice and C is a sum of rectangular bands;*

b) *$S\cong\mathcal{M}^0(D; I, \Lambda; P)$, where $D=D^1$ is a bisimple inverse semigroup and E_S is a semigroup;*

c) *$S\cong\mathcal{M}^0(D; I, \Lambda; P)$, where D is as in* b) *and P satisfies (N);*

d) *$S\cong D\times^0 B$, where D is as in* b) *and B is a rectangular* 0*-band whose idempotents form a subsemigroup.*

Proof. a)$\Rightarrow$b). Let B as in the proof of 5.3, b)$\Rightarrow$c), and let $A=T\times^0 B$. Then $E_B\cong C$ so that $E_S\cong T\times^0 C\cong T\times^0 E_B\cong E_A$. Since B is a rectangular 0-band, B satisfies 5.1 a). It follows that in turn A, E_A, E_S, S satisfy 5.1 a); the last implication holds since S is regular. If θ is a semigroup isomorphism of E_S onto E_A, then letting $\varphi=\psi=\theta$, all conditions in 5.1 b) are trivially satisfied. Let $e,f\in E_S$ and suppose $eS\cap fS\neq 0$. Then $ex=fy\neq 0$ for some $x, y\in S$. Let x' be an inverse of x and w be an inverse of yx'. Using the fact that idempotents of S form a subsemigroup, we obtain

$$f(yx'w)=e(xx')w=e(xx')e(xx')w=e(xx')f(yx'w)\neq 0,$$

which implies $eE_S\cap fE_S\neq 0$. We identify E_S with $T\times^0 C$ and write $e=(a, u)$, $f=(b, v)$, so that $(a, u)(s, t)=(b, v)(p, q)\neq 0$ for some $(s, t), (p, q)\in T\times^0 C$. It follows that $as=bp$, $ut=vq\neq 0$. Since C is a sum of rectangular bands, we have $u, t, v, q\in B_\alpha$ for some rectangular band B_α. From $ut=vq=uvq$ it then follows $uv=u(vqv)=(uvqv)=vqv\in vC$. In T, trivially $ab\in bT$, so that $ef\in fE_S\subseteq fS$. Thus 5.2 (b) holds and hence also 5.1 c) i); c) ii) is verified analogously. By 5.1, b) holds.

b)$\Rightarrow$c). If $p_{\lambda i}\neq 0$, $p_{\lambda j}\neq 0$, $p_{\mu j}\neq 0$, then

$$(p_{\lambda i}^{-1}; i, \lambda)(p_{\mu j}^{-1}, j, \mu) = (p_{\mu i}^{-1}; i, \mu)$$

since the product on the left is a nonzero idempotent. Hence $p_{\lambda i}^{-1}p_{\lambda j}p_{\mu j}^{-1}=p_{\mu i}^{-1}$ and (N) holds.

c)$\Rightarrow$d). It is easy to see that (N) implies (M). Thus by 4. 3, $S\cong D\times^0 B$, $S_1\cong G\times^0 B$, where D, B, G are as in 4. 3. Moreover, $S_1=\mathcal{M}^0(G; I, \Lambda; P)$, where P satisfies (N). From the proof of b)$\Rightarrow$c), it follows directly that E_{S_1} is a semigroup and since $S_1\cong G\times^0 B$, E_B also is a semigroup.

d)$\Rightarrow$a). We identify S with $D\times^0 B$. Since D is bisimple and B is 0-bisimple, it follows easily that S is 0-bisimple. Evidently $E_S=E_D\times^0 E_B$, where E_D is a semilattice with identity and E_B is a sum of rectangular bands by 5. 3.

Corollary 5. 6. *The following statements are equivalent for any semigroup S:*

a) *S is bisimple and $E_S\cong T\times B$, where $T=T^1$ is a semilattice and B is a rectangular band;*

b) *$S\cong\mathcal{M}(D; I, \Lambda; P)$, where $D=D^1$ is a bisimple inverse semigroup and E_S is a semigroup;*

c) *$S\cong\mathcal{M}(D; I, \Lambda; P)$, where D is as in* b) *and P satisfies (N);*

d) *$S\cong D\times B$, where D is as in* b) *and B is a rectangular band.*

Recall that an inverse rectangular 0-band is called a Brandt 0-band (3. 2, [3]). A semigroup K which is a sum of rectangular bands each of which contains only one element is characterized by the fact that K has 0 and at least one more element, and for any $a, b\in K$:

$$ab=\begin{cases} a & \text{if } a=b; \\ 0 & \text{if } a\neq b; \end{cases}$$

call such a semigroup a *Kronecker semigroup.*

Corollary 5. 7. *The following statements are equivalent for any semigroup S:*

a) *S is 0-bisimple and $E_S\cong T\times^0 K$, where $T=T^1$ is a semilattice and K is a Kronecker semigroup;*

b) *$S\cong\mathcal{M}^0(D; I, \Lambda; P)$, where $D=D^1$ is a bisimple inverse semigroup and E_S is a semilattice;*

c) *$S\cong\mathcal{M}^0(D; I, I; \Delta)$; where D is as in* b) *and Δ is the $I\times I$-unit matrix;*

d) *$S\cong D\times^0 B$, where D is as in* b) *and B is a Brandt 0-band.*

The proof of 5. 6 and 5. 7 follows easily from 5. 5 and is omitted. Note that further characterizations of semigroups appearing in these corollaries can be given using the results of the previous section, i.e., using the notions of a Rees 0-composition and of a matrix of semigroups.

6. Example and conclusions

The following example shows that a bisimple regular semigroup need not be a matrix of bisimple inverse semigroups. Let S be the semigroup generated by a and b subject to the relations $a=aba$, $b=bab=ab^2$. The elements of S can be written in an array:

$$\underbrace{\begin{matrix} a & a^2 & \cdots & a^m & \cdots \\ ba & ba^2 & \cdots & ba^m & \cdots \\ \vdots & \vdots & & \vdots & \\ b^n a & b^n a^2 & \cdots & b^n a^m & \cdots \\ \vdots & \vdots & & \vdots & \end{matrix}}_{L_1} \qquad \underbrace{\begin{matrix} ab & a^2 b & \cdots & a^m b & \cdots \\ b & ba^2 b & \cdots & ba^m b & \cdots \\ \vdots & \vdots & & \vdots & \\ b^n & b^n a^2 b & \cdots & b^n a^m b & \cdots \\ \vdots & \vdots & & \vdots & \end{matrix}}_{L_2}$$

The $\mathscr{R}$-classes constitute the rows and the $\mathscr{L}$-classes the columns of this array. Hence S is bisimple and regular. E_S consists of two descending chains

$$ba>b^2a^2>\cdots>b^m a^m>\cdots, \qquad ab>ba^2b>\ldots>b^m a^{m+1}b\cdots>\cdots,$$

no two elements belonging to different chains are comparable, and E_S is a subsemigroup of S. Both L_1 and L_2 are left ideals and the partition induced is the maximal matrix decomposition of S ([4]). Since L_1 is not regular, S is not a matrix of inverse semigroups. L_1 is the subsemigroup of the bicyclic semigroup generated by $p_1=a$, $p_2=b$ obtained by omitting the $\mathscr{L}$-class of the identity; L_2 is the bicyclic semigroup generated by $p_2=a^2b$, $q_2=b$.

That S is not a matrix of bisimple inverse semigroups with identity can also be (more easily) deduced from our results. For suppose it is; then by 2. 3, 3. 10, and 3. 2, $S\cong\mathscr{M}(D; I, \Lambda; P)$, where $D=D^1$ is a bisimple inverse semigroup. Since E_S is a semigroup, by 5. 6 we must have $E_S\cong T\times B$, where $T=T^1$ is a semilattice and B is a rectangular band. Since E_S consists of two chains, we must have $|B|=2$, and the set $\{ba, ab\}$ must be either a left or a right zero semigroup. However, $(ba)(ab)=ba^2b\notin\{ba, ab\}$.

Using the theory developed in the previous section, whenever the structure of a class of bisimple inverse semigroups with identity is described by means of some construction involving the group of units, the structure of 0-bisimple semigroups which are 0-matrices of these semigroups is readily available. For example, if S is a 0-bisimple (or, equivalently, regular; 3. 11) semigroup which is a 0-matrix of bisimple ω-semigroups introduced by REILLY [6], then $S\cong\mathscr{M}^0(D; I; \Lambda, P)$, where D is a bisimple ω-semigroup. In fact, S can be represented as the set $(G\times N\times N\times I\times\Lambda)\cup 0$, where G is a group, N is the set of nonnegative integers, with multiplication

$$(g,m,n,i,\lambda)(h,p,q,j,\mu) = (g\alpha^{p-r}q_{\lambda j}\alpha^{n+p-r}h\alpha^{n-r}, m+p-r, n+q-r, i, \mu)$$

if $q_{\lambda j} \neq 0$, otherwise equal to zero, where $Q = (q_{\lambda i})$ is a regular $\Lambda \times I$-matrix over G°, α is a fixed endomorphism of G, α^t is the t-th iterate of α, with α° the identity transformation, and $r = \min\{n, p\}$. S can also be characterized by using 5. 1 (for special cases, see 5. 5, 5. 6, and 5. 7).

The case of a matrix of semigroups (or an r-composition, section 2) as treated in section 3, serves the same purpose as described above for bisimple inverse semigroups with identity, for a much larger class of semigroups (composable semigroups, see examples in section 2).

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On semigroups which are semilattices of groups

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Let S be a semigroup. [1]) We shall say that S has property (M) if the relation

$$L \cap R = LR \tag{1}$$

holds for each left ideal L and for each right ideal R of S. Furthermore we say that S has property (L) resp. (R) if the relation

$$L_1 \cap L_2 = L_1 L_2 \tag{2}$$

resp.

$$R_1 \cap R_2 = R_1 R_2 \tag{3}$$

holds for every pair of left and right ideals of S, respectively.

Recently the author proved that a semigroup S having both properties (L) and (R) is a disjoint union of groups. In this note we prove that a semigroup S with property (M) is a semilattice of groups. It may also be proved that a semigroup having both properties (L) and (R) is a semilattice of groups.

First we prove some results concerning a semigroup having the property (M).

Theorem 1. *In a semigroup S having property (M) every one-sided ideal is a two-sided ideal.*

Proof. Let S be a semigroup with property (M), and suppose that L is an arbitrary left ideal of S. Then by (1) we have

$$L = L \cap S = LS$$

that is, L is also a right ideal of S. Therefore L is a two-sided ideal of S, as we stated. Similarly, any right ideal R of S is also a two-sided ideal, because of

$$R = S \cap R = SR.$$

Theorem 2. *Any semigroup with property (M) is a normal semigroup.*

[1]) We adopt the terminology of CLIFFORD and PRESTON [2]. See also LJAPIN [6] and SZÁSZ [8].

Proof. Let S be a semigroup having the property (M), and a be an arbitrary element of S. Then (1) implies that

$$aS = S \cap aS = SaS. \tag{4}$$

Similarly,

$$Sa = Sa \cap S = SaS. \tag{5}$$

The relations (4) and (5) imply

$$aS = Sa \tag{6}$$

for each element a in S, i.e., S is a normal semigroup. (See [7]).

Theorem 3. *Any semigroup S having property (M) is regular.* [2])

Proof. Let S be a semigroup with property (M). In a recent note [4] the author proved that a normal semigroup is regular if and only if every left ideal L of S is idempotent, i.e. $L^2 = L$. By Theorem 2, S is normal. If A is a left ideal of S then by Theorem 1, A is a two-sided ideal of S, and the relation (1) implies that

$$A = AA$$

in case of $L = R = A$. Therefore any ideal of S is idempotent, which proves the regularity of S.

Now we are ready to prove the following result:

Theorem 4. *Any semigroup S with property (M) is a semilattice of groups.*

Proof. Let S be a semigroup having the property (M). First we show that for every element a of S

$$a \in Sa^2 \cap a^2 S \tag{7}$$

holds. By Theorem 3 we have $a \in aSa$. But by Theorem 2, the semigroup S is normal, i.e. $aS = Sa$ for each element a in S. Hence $aSa = a^2 S = Sa^2$. Therefore the inclusion (7) follows.

Secondly we show that the idempotent elements of S commute, that is, if e and f are idempotent elements of S, then

$$ef = fe \tag{8}$$

holds. But this is an easy consequence of a result due to Schwarz [7] in virtue of which the idempotent elements of a normal semigroup lie in the center.

Now (7) and (8) imply that S is a semilattice of groups, by a result of Croisot [3] (see also Clifford [1], Theorem 8).

The following result may be proved analogously:

[2]) More generally, *a semigroup S having property (M) is an inverse semigroup.* This follows from Theorems 2 and 3.

Theorem 5. *Suppose that the semigroup S has properties (L) and (R). Then S is a semilattice of groups.*

Corollary. *If S is a semigroup having both properties (L) and (R) then S is a union of disjoint groups.*

This is an easy consequence of Theorem 5, and was recently proved by the author in [5].

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Erweiterung von Halbgruppen durch wiederholte Quotientenbildung. I

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Einleitung

Ist $\mathfrak{N}$ eine Halbgruppe und $\mathfrak{n}$ eine Unterhalbgruppe (zweiseitig) regulärer Elemente von $\mathfrak{N}$, so versteht man unter einer Rechtsquotientenhalbgruppe $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ von $\mathfrak{N}$ nach $\mathfrak{n}$ eine Oberhalbgruppe von $\mathfrak{N}$ mit Einselement, in der jedes Element $\alpha \in \mathfrak{n}$ ein Inverses α^{-1} besitzt und deren Elemente als Rechtsquotienten $a\alpha^{-1}$ mit $a \in \mathfrak{N}$ und $\alpha \in \mathfrak{n}$ dargestellt werden können. Letzteres drücken wir auch durch die Schreibweise $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}) = \mathfrak{N}\mathfrak{n}^{-1}$ aus und nennen $\mathfrak{n}$ eine rechtsseitige Nennermenge von $\mathfrak{N}$.

Eine solche Rechtsquotientenhalbgruppe $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ existiert nach [1], [2] genau dann, wenn die folgende Bedingung $Q_r(\mathfrak{N}, \mathfrak{n})$ erfüllt ist: Zu je zwei Elementen $a \in \mathfrak{N}$ und $\alpha \in \mathfrak{n}$ gibt es Elemente $l \in \mathfrak{N}$ und $\lambda \in \mathfrak{n}$ mit $a\lambda = \alpha l$. Die Rechtsquotientenhalbgruppe $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ ist dann durch $\mathfrak{N}$ und $\mathfrak{n}$ bis auf Isomorphie eindeutig bestimmt.

Dagegen können verschiedene Unterhalbgruppen $\mathfrak{n}_i$ $(i \in I)$ von $\mathfrak{N}$ zur gleichen Rechtsquotientenhalbgruppe $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}_i)$ führen. Unter ihnen gibt es dann genau eine maximale Halbgruppe $\mathfrak{n}$, die gerade aus allen in $\mathfrak{S}$ invertierbaren Elementen von $\mathfrak{N}$ besteht. Nach [2] (§ 4, Satz 2) ist eine rechtsseitige Nennermenge $\mathfrak{n}$ von $\mathfrak{N}$ genau dann relativ maximal in diesem Sinne, wenn aus $ab \in \mathfrak{n}$ für beliebige reguläre Elemente a und b aus $\mathfrak{N}$ stets $a \in \mathfrak{n}$ und $b \in \mathfrak{n}$ folgt. Dabei erhält man aus jeder Nennermenge $\mathfrak{n}_i$ mit $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}_i)$ die zugehörige relativ maximale Nennermenge $\mathfrak{n}$ mit $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ durch Hinzunahme aller regulären Elemente a und b aus $\mathfrak{N}$ mit $ab \in \mathfrak{n}_i$.

Andererseits sprechen wir von der absolut maximalen rechtsseitigen Nennermenge von $\mathfrak{N}$ in folgendem Sinne: Wenn nämlich die Halbgruppe $\mathfrak{N}$ überhaupt Rechtsquotientenhalbgruppen $\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ besitzt, so liegen alle rechtsseitigen Nennermengen $\mathfrak{n}$ in einer maximalen Halbgruppe $\mathfrak{m}$ dieser Art (vgl. [2], § 4, Satz 3). Es gibt dann also die bis auf Isomorphie eindeutig bestimmte maximale Rechtsquotientenhalbgruppe $\mathfrak{Q}_r(\mathfrak{N}) = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{m})$, die jede Rechtsquotientenhalbgruppe $\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ von $\mathfrak{N}$ als Unterstruktur enthält.

Schließlich ist für unsere nachfolgenden Untersuchungen noch von Bedeutung, daß sich jeweils endlich viele Elemente einer Rechtsquotientenhalbgruppe $\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ stets mit „gleichem Nenner", also in der Form $a_1\alpha^{-1}, \cdots, a_n\alpha^{-1}$ schreiben lassen. Natürlich gelten alle diese Resultate in entsprechender Form auch für Linksquotientenhalbgruppen $\mathfrak{Q}_l(\mathfrak{N}, \mathfrak{n})$, wobei wir hier wie im Folgenden mit jeder Begriffsbildung bzw. Aussage auch immer die aus ihr durch „Vertauschung von rechts und links" hervorgehende duale Begriffsbildung bzw. Aussage als gegeben ansehen. Wir erwähnen in diesem Zusammenhang noch, daß für eine Unterhalbgruppe $\mathfrak{n}$ von $\mathfrak{N}$, die sowohl rechtsseitige wie linksseitige Nennermenge von $\mathfrak{N}$ ist, die zugehörigen Quotientenhalbgruppen $\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ und $\mathfrak{Q}_l(\mathfrak{N}, \mathfrak{n})$ als gleich angesehen werden können (vgl. [2]).

In der vorliegenden Arbeit untersuchen wir nun Halbgruppenerweiterungen, die sich durch Nacheinanderanwendung von Quotientenbildungen der eben beschriebenen Art ergeben. Diese Problemstellung wurde in [3] aufgeworfen und zunächst gezeigt, daß zwei „gleichseitige" Erweiterungsschritte zu nichts Neuem führen: Ist nämlich $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ eine Rechtsquotientenhalbgruppe von $\mathfrak{N}$ nach $\mathfrak{n}$ und $\mathfrak{T} = \mathfrak{Q}_r(\mathfrak{S}, \mathfrak{s})$ eine Rechtsquotientenhalbgruppe von $\mathfrak{S}$ nach einer Unterhalbgruppe $\mathfrak{s}$ von $\mathfrak{S}$, so ist $\mathfrak{T}$ auch schon als Rechtsquotientenhalbgruppe $\mathfrak{T} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{t})$ von $\mathfrak{N}$ nach einer geeigneten Unterhalbgruppe $\mathfrak{t}$ von $\mathfrak{N}$ zu gewinnen. Damit erhebt sich als nächstes die Frage, ob etwa auch die Nacheinanderanwendung einer Rechtsquotientenerweiterung $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ und einer Linksquotientenerweiterung $\mathfrak{T} = \mathfrak{Q}_l(\mathfrak{S}, \mathfrak{s})$ stets durch einen Schritt ersetzt werden kann, also $\mathfrak{T}$ selbst schon Rechts- oder Linksquotientenhalbgruppe von $\mathfrak{N}$ ist. Dies ist jedoch nicht der Fall, wie wir zunächst durch ein Beispiel in § 1 nachweisen.

Im folgenden Paragraphen wenden wir uns einer allgemeinen Theorie der Erweiterung von Halbgruppen durch abwechselnde Bildung von Rechts- und Linksquotientenhalbgruppen zu. Beginnend mit einer Rechtsquotientenhalbgruppe von $\mathfrak{N}$ sprechen wir nach k Schritten von einer k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_k)$ von $\mathfrak{N}$, wobei $\mathfrak{n}_\varkappa$ ($\varkappa = 1, 2, \ldots, k$) jeweils die im $\varkappa$-ten Schritt verwendete Nennermenge bezeichnet (vgl. Definition 1). Über diese Halbgruppen $\mathfrak{n}_\varkappa$ können wir noch in gewisser Weise verfügen, ohnedabei die einzelnen Quotientenerweiterungsschritte abzuändern; insbesondere zeigen wir, daß wir ohne Beschränkung der Allgemeinheit stets $\mathfrak{n}_1 \subseteqq \mathfrak{n}_2 \subseteqq \cdots \subseteqq \mathfrak{n}_k$ annehmen dürfen. Da jedoch die Nennermengen $\mathfrak{n}_\varkappa$ jeweils erst nach der $(\varkappa-1)$-ten Quotientenerweiterung von $\mathfrak{N}$ als Unterhalbgruppe von $\mathfrak{N}_{\varkappa-1} = \mathfrak{Q}_r^{\varkappa-1}(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_{\varkappa-1})$ zur Verfügungstehen, gehen wir zu ihren Durchschnitten $\mathfrak{x}_\varkappa = \mathfrak{n}_\varkappa \cap \mathfrak{N}$ mit der Halbgruppe $\mathfrak{N}$ über und versuchen, mit ihrer Hilfe Aussagen über k-te r-Quotientenhalbgruppen $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$ von $\mathfrak{N}$ zu gewinnen. Eine so entstehende Unterhalbgruppenkette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ von $\mathfrak{N}$ nennen wir eine Q_r-Kette von $\mathfrak{N}$ der Länge k. Wir zeigen, daß dann $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$ auch schon durch diese Unterhalbgruppen von $\mathfrak{N}$ eindeutig bis auf Isomorphie bestimmt ist (Satz 2), was die Bezeichnung $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ rechtfertigt. Dabei spielt, wie überhaupt bei allen Überlegungen, eine wichtige Rolle, daß die k-te r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ von $\mathfrak{N}$ zugleich eine $(k-1)$-te l-Quotientenhalbgruppe $\mathfrak{Q}_l^{k-1}(\mathfrak{N}_1; \mathfrak{y}_2, \cdots, \mathfrak{y}_k)$ von $\mathfrak{N}_1 = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{x}_1)$ ist, wobei die Unterhalbgruppen der Q_l-Kette $\mathfrak{y}_2 \subseteqq \mathfrak{y}_3 \subseteqq \cdots \subseteqq \mathfrak{y}_k$ von $\mathfrak{N}_1$ als Rechtsquotientenhalbgruppen $\mathfrak{y}_\varkappa = \mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1)$ gewählt werden können (Satz 1). In dem Hauptsatz (Satz 3) dieser Arbeit geben wir dann notwendige und hinreichende (bereits in $\mathfrak{N}$ nachprüfbare) Bedingungen dafür an, daß eine Kette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ von Unterhalbgruppen regulärer Elemente von $\mathfrak{N}$ eine Q_r-Kette ist, also die k-te r-Quotientenhalbgruppe $\mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ existiert.

Eine wichtige Ergänzung zu diesem Resultat stellt ein zweiter Hauptsatz (Satz 6) dar, den wir jedoch erst in der Fortsetzung dieser Arbeit behandeln werden. Wir werden dort nämlich zeigen, daß es für jede natürliche Zahl k Halbgruppen $\mathfrak{N}$ gibt, zu denen eine k-te r-Quotientenhalbgruppe $\mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ existiert, die jedoch auf keine Weise in weniger als k Schritten durch Quotientenerweiterung von $\mathfrak{N}$ gewonnen werden kann.

Dagegen behandeln wir in dem vorliegenden Teil I, § 3 noch einige Fragen der Darstellung der Elemente einer k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$. Wir zeigen (Satz 4), daß für jedes Element $a_k \in \mathfrak{N}_k$ eine Quotientendarstellung der Form

$$a_k = x_1 x_2^{-1} \cdots x_{k-1}^{-1} \cdot a x_k^{-1} \cdot x_{k-1} \cdots x_2 x_1^{-1}$$

bzw.

$$a_k = x_1 x_2^{-1} \cdots x_{k-1} \cdot x_k^{-1} a \cdot x_{k-1}^{-1} \cdots x_2 x_1^{-1}$$

möglich ist, je nachdem ob k eine ungerade oder eine gerade Zahl ist. Zur Darstellung eines Elementes von $\mathfrak{N}_k$ braucht also jeweils nur ein Element aus jeder Nennermenge $\mathfrak{x}_\varkappa$ verwendet zu werden. Darüber hinaus weisen wir nach (Satz 5), daß sogar je endlich viele Elemente von $\mathfrak{N}_k$ eine solche Darstellung mit „den gleichen Nennern" gestatten, wobei also für jedes Element die gleichen $x_\varkappa \in \mathfrak{x}_\varkappa$ $(\varkappa = 1, \cdots, k)$ auftreten.

Wir vermerken noch, daß alle unseren allgemeinen Überlegungen auch für die Erweiterung von Halbringen durch wiederholte Quotientenbildung zutreffen. Gemäß [2], § 5 ist nämlich jeder Rechtsquotientenhalbring $\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ eines Halbringes $\mathfrak{N}$ nach einer Unterhalbgruppe $\mathfrak{n}$ multiplikativ regulärer Elemente von $\mathfrak{N}$ bereits durch die Halbgruppenerweiterung $\mathfrak{Q}_r(\mathfrak{N}^\times, \mathfrak{n})$ der multiplikativen Halbgruppe $\mathfrak{N}^\times$ von $\mathfrak{N}$ eindeutig festgelegt, da sich die Addition von $\mathfrak{N}$ stets auf eine und nur eine Weise zu einer Addition in jeder Rechtsquotientenhalbgruppe $\mathfrak{Q}_r(\mathfrak{N}^\times, \mathfrak{n})$ fortsetzen läßt. Definieren wir nun einen k-ten r-Quotientenhalbring $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$ eines Halbringes $\mathfrak{N}$ nach den jeweils im $\varkappa$-ten Schritt als Nennermengen auftretenden Unterhalbgruppen $\mathfrak{n}_\varkappa$ genau wie bei Halbgruppen, so läßt sich diese

Aussage auf jeden Teilschritt anwenden, womit alle Ergebnisse der Paragraphen 2 und 3 ebenso für die Erweiterung von Halbringen durch Quotientenbildung gelten. Dabei ermöglicht es die in Satz 5 gezeigte Darstellungsmöglichkeit der Elemente von $\mathfrak{N}_k$, die Addition in $\mathfrak{N}_k$ sogar direkt auf die Addition in $\mathfrak{N}$ zurückzuführen. Auf die Existenz k-ter r-Quotientenhalbringe werden wir in einer späteren Arbeit zurückkommen.

§ 1

Im Folgenden konstruieren wir eine Halbgruppe $\mathfrak{N}$, für die eine Linksquotientenhalbgruppe einer Rechtsquotientenhalbgruppe von $\mathfrak{N}$ existiert, ohne daß eine solche Oberstruktur durch eine einmalige Quotientenerweiterung erreichbar ist. Wir werden $\mathfrak{N}$ als direktes Produkt zweier geeigneter Halbgruppen gewinnen und stellen deshalb der eigentlichen Konstruktion den folgenden Hilfssatz voran. Dabei verstehen wir unter dem direkten Produkt $\mathfrak{N}=\mathfrak{N}_1\otimes\mathfrak{N}_2$ zweier Halbgruppen $\mathfrak{N}_1$, $\mathfrak{N}_2$ zunächst die Produktmenge $\mathfrak{N}_1\times\mathfrak{N}_2$ mit komponentenweiser Multiplikation; da wir jedoch für beide Halbgruppen $\mathfrak{N}_i$ die Existenz eines Einselementes voraussetzen, dürfen wir die Elemente von $\mathfrak{N}$ in der Form $n=n_1n_2$ mit $n_i\in\mathfrak{N}_i$ schreiben.

Hilfssatz 1. *Sind $\mathfrak{N}_1$ und $\mathfrak{N}_2$ Halbgruppen mit Einselement und den absolut maximalen rechtsseitigen Nennermengen $\mathfrak{n}_1$ bzw. $\mathfrak{n}_2$, so besitzt das direkte Produkt $\mathfrak{N}=\mathfrak{N}_1\otimes\mathfrak{N}_2$ die Halbgruppe $\mathfrak{n}=\mathfrak{n}_1\otimes\mathfrak{n}_2$ als absolut maximale rechtsseitige Nennermenge.*

Beweis. Zunächst ist $\mathfrak{n}=\mathfrak{n}_1\otimes\mathfrak{n}_2$ jedenfalls eine Unterhalbgruppe regulärer Elemente von $\mathfrak{N}$. Zum Beweis der Bedingung $Q_r(\mathfrak{N},\mathfrak{n})$ wählen wir a_1a_2 beliebig aus $\mathfrak{N}$ und $\alpha_1\alpha_2$ beliebig aus $\mathfrak{n}$. Wegen $Q_r(\mathfrak{N}_1,\mathfrak{n}_1)$ und $Q_r(\mathfrak{N}_2,\mathfrak{n}_2)$ gilt dann $a_1\lambda_1=\alpha_1l_1$ und $a_2\lambda_2=\alpha_2l_2$ mit geeigneten Elementen $l_1\in\mathfrak{N}_1$, $l_2\in\mathfrak{N}_2$, $\lambda_1\in\mathfrak{n}_1$ und $\lambda_2\in\mathfrak{n}_2$, und die Elemente $l_1l_2\in\mathfrak{N}$ und $\lambda_1\lambda_2\in\mathfrak{n}$ erfüllen

$$a_1a_2\cdot\lambda_1\lambda_2=a_1\lambda_1a_2\lambda_2=\alpha_1l_1\alpha_2l_2=\alpha_1\alpha_2\cdot l_1l_2,$$

womit $\mathfrak{n}$ als rechtsseitige Nennermenge von $\mathfrak{N}$ nachgewiesen ist. Es existiert also auch die absolut maximale rechtsseitige Nennermenge $\mathfrak{m}$ von $\mathfrak{N}$, deren Elemente die Darstellung $\alpha=\alpha_1\alpha_2$ mit $\alpha_i\in\mathfrak{m}_i\subseteqq\mathfrak{N}_i$ $(i=1,2)$ gestatten, wobei $\mathfrak{m}_i$ die Gesamtheit aller Komponenten α_i der Elemente α von $\mathfrak{m}$ bezeichnet. Dabei ist klar, daß $\mathfrak{m}_i$ Unterhalbgruppe regulärer Elemente von $\mathfrak{N}_i$ ist, für die wegen $\mathfrak{m}\supseteqq\mathfrak{n}=\mathfrak{n}_1\otimes\mathfrak{n}_2$ sofort $\mathfrak{m}_i\supseteqq\mathfrak{n}_i$ folgt. Andererseits gibt es wegen $Q_r(\mathfrak{N},\mathfrak{m})$ zu jedem $a_1a_2\in\mathfrak{N}$ und zu jedem $\alpha_1\alpha_2\in\mathfrak{m}$ Elemente $l_1l_2\in\mathfrak{N}$ und $\lambda_1\lambda_2\in\mathfrak{m}$ mit $a_1a_2\lambda_1\lambda_2=\alpha_1\alpha_2l_1l_2$, also $a_i\lambda_i=\alpha_il_i$ für $i=1,2$. Das bedeutet aber gerade die Gültigkeit von $Q_r(\mathfrak{N}_i,\mathfrak{m}_i)$, womit aus der absoluten Maximalität von $\mathfrak{n}_i$ schließlich auch $\mathfrak{m}_i\subseteqq\mathfrak{n}_i$ folgt.

Wir betrachten nun eine Halbgruppe $\mathfrak{H}$ mit Einselement, welche von zwei

Elementen A und B mit der Relation $A^2B=BA$ erzeugt wird, weswegen sich ihre Elemente jedenfalls in der Form A^xB^y (x, y nicht negative ganze Zahlen) angeben lassen und je zwei Elemente von $\mathfrak{H}$ nach dei Regel

$$A^{x_1}B^{y_1}\cdot A^{x_2}B^{y_2} = A^{x_1+2^{y_1}x_2}B^{y_1+y_2}$$

multipliziert werden. Man kann nun nachprüfen, daß die oben angegebene Darstellung der Elemente von $\mathfrak{H}$ eindeutig und daß $\mathfrak{H}$ eine reguläre Halbgruppe ist, worauf wir jedoch an dieser Stelle verzichten wollen, zumal sich beides aus allgemeineren Überlegungen im zweiten Teil mit ergeben wird. Für diese Halbgruppe $\mathfrak{H}$ zeigen wir zunächst:

a) Die absolut maximale rechtsseitige Nennermenge von $\mathfrak{H}$ ist $\mathfrak{n}_1=\{A^x\}$;

b) die absolut maximale linksseitige Nennermenge von $\mathfrak{H}$ ist $\mathfrak{H}$ selbst.

Vorerst ist klar, daß $\mathfrak{n}_1$ eine Unterhalbgruppe regulärer Elemente von $\mathfrak{H}$ ist. Sind weiterhin $a=A^xB^y$ und $\alpha=A^\xi$ beliebige Elemente von $\mathfrak{H}$ bzw. $\mathfrak{n}_1$, dann erfüllen die Elemente

$$l = A^{x+\xi(2^y-1)}B^y\in\mathfrak{H} \quad \text{und} \quad \lambda = A^\xi\in\mathfrak{n}_1$$

die Gleichung

$$a\lambda = A^xB^y\cdot A^\xi = A^{x+2^y\xi}B^y = A^\xi\cdot A^{x+\xi(2^y-1)}B^y = \alpha l,$$

also gilt $Q_r(\mathfrak{H},\mathfrak{n}_1)$. Dagegen erfüllt keine $\mathfrak{n}_1$ echt umfassende Unterhalbgruppe $\bar{\mathfrak{n}}_1$ von $\mathfrak{H}$ die Bedingung $Q_r(\mathfrak{H},\bar{\mathfrak{n}}_1)$. Es enthält nämlich $\bar{\mathfrak{n}}_1$ wenigstens ein Element $\bar{\alpha}=A^\xi B^\eta$ mit $\eta\neq0$. Wählen wir dann etwa $a=A^xB^y$ aus $\mathfrak{H}$ mit $y\neq0$ und $x\not\equiv\xi \bmod 2$, so führt jedes $\bar{\lambda}=A^{\xi'}B^{\eta'}\in\bar{\mathfrak{n}}_1$ und jedes $l=A^{x'}B^{y'}\in\mathfrak{H}$ auf

$$a\bar{\lambda} = A^{x+2^y\xi'}B^{y+\eta'} \quad \text{und} \quad \bar{\alpha}l = A^{\xi+2^\eta x'}B^{\eta+y'},$$

was aber wegen $x\not\equiv\xi \bmod 2$ und $y\neq0$, $\eta\neq0$ auf $a\bar{\lambda}\neq\bar{\alpha}l$ hinausläuft.

Für b) genügt es schließlich zu zeigen, daß $Q_l(\mathfrak{H},\mathfrak{H})$ erfüllt ist. Mit den beliebig vorgegebenen Elementen $a=A^xB^y$ und $\alpha=A^\xi B^\eta$ aus $\mathfrak{H}$ erfüllen gerade $l=A^{2^\eta x}B^y$ und $\lambda=A^{2^y\xi}B^\eta$ aus $\mathfrak{H}$ die Gleichung

$$\lambda a = A^{2^y\xi}B^\eta\cdot A^xB^y = A^{2^y\xi+2^\eta x}B^{\eta+y} = A^{2^\eta x}B^y\cdot A^\xi B^\eta = l\alpha.$$

Entsprechend besteht die von zwei Elementen C und D mit der Relation $DC=CD^2$ erzeugte Halbgruppe $\mathfrak{F}$ mit Einselement gerade aus den Elementen C^zD^w (z, w nicht negative ganze Zahlen), die vermöge $A^xB^y\to C^yD^x$ zur Halbgruppe $\mathfrak{H}$ antiisomorph ist. Daraus folgt für $\mathfrak{F}$:

a) Die absolut maximale linksseitige Nennermenge von $\mathfrak{F}$ ist $\mathfrak{n}_2=\{D^w\}$;

b) die absolut maximale rechtsseitige Nennermenge von $\mathfrak{F}$ ist $\mathfrak{F}$ selbst.

Gemäß Hilfssatz 1 und seiner dualen Aussage gilt dann für das direkte Produkt $\mathfrak{R}=\mathfrak{H}\otimes\mathfrak{F}$ von $\mathfrak{H}$ und $\mathfrak{F}$:

A) Die absolut maximale rechtsseitige Nennermenge von $\mathfrak{R}$ ist $\mathfrak{n}=\mathfrak{n}_1\otimes\mathfrak{F}\neq\mathfrak{R}$;

B) die absolut maximale linksseitige Nennermenge von $\mathfrak{R}$ ist $\mathfrak{m}=\mathfrak{H}\otimes\mathfrak{n}_2\neq\mathfrak{R}$.

Wir betrachten nun die Rechtsquotientenhalbgruppe $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}) = (\mathfrak{H} \otimes \mathfrak{F})(\mathfrak{n}_1 \otimes \mathfrak{F})^{-1}$, deren Elemente also die Darstellung $a = A^x B^y C^z D^w D^{-\omega} C^{-\zeta} A^{-\xi}$ gestatten, und zeigen, daß die Linksquotientenhalbgruppe $\mathfrak{T} = \mathfrak{Q}_l(\mathfrak{S}, \mathfrak{b})$ von $\mathfrak{S}$ nach der von B erzeugten Unterhalbgruppe $\mathfrak{b} = \{B^y\}$ existiert. Zunächst ist mit $\mathfrak{H}$ und $\mathfrak{F}$ auch $\mathfrak{N} = \mathfrak{H} \otimes \mathfrak{F}$ und damit gemäß [2], S. 212 auch $\mathfrak{S}$ regulär, so daß wir nur noch $Q_l(\mathfrak{S}, \mathfrak{b})$ nachzuweisen haben. Dabei verwenden wir, daß die Elemente A^{-1}, C^{-1} und D^{-1} von $\mathfrak{S}$ ersichtlich den Beziehungen $BA^{-1} = A^{-2}B$, $C^{-1}B = BC^{-1}$ bzw. $D^{-1}B = BD^{-1}$ genügen, und betrachten beliebige Elemente $a = A^x B^y C^z D^w D^{-\omega} C^{-\zeta} A^{-\xi}$ aus $\mathfrak{S}$ und $\alpha = B^\eta$ aus $\mathfrak{b}$. Mit den Elementen

$$l = A^{2^\eta x} B^y C^z D^w D^{-\omega} C^{-\zeta} A^{-2^\eta \xi} \in \mathfrak{S} \quad \text{und} \quad \lambda = B^\eta \in \mathfrak{b}$$

gilt dann

$$\lambda a = B^\eta \cdot A^x B^y C^z D^w D^{-\omega} C^{-\zeta} A^{-\xi} = A^{2^\eta x} B^{\eta+y} C^z D^w D^{-\omega} C^{-\zeta} A^{-\xi}$$

$$l\alpha = A^{2^\eta x} B^y C^z D^w D^{-\omega} C^{-\zeta} A^{-2^\eta \xi} \cdot B^\eta = A^{2^\eta x} B^{y+\eta} C^z D^w D^{-\omega} C^{-\zeta} A^{-\xi}$$

also gerade $\lambda a = l\alpha$. In $\mathfrak{T} = \mathfrak{Q}_l(\mathfrak{S}, \mathfrak{b}) = \mathfrak{Q}_l(\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}), \mathfrak{b})$ sind damit alle Elemente von $\mathfrak{N}$ invertierbar, was wegen A) und B) weder für die maximale Rechtsquotientenhalbgruppe $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ noch für die maximale Linksquotientenhalbgruppe $\mathfrak{S}' = \mathfrak{Q}_l(\mathfrak{N}, \mathfrak{m})$ von $\mathfrak{N}$ zutrifft.

§ 2

Zur formalen Vereinfachung unserer allgemeinen Untersuchungen stellen wir zunächst folgenden Hilfssatz bereit:

Hilfssatz 2. *Es sei $\mathfrak{N}$ eine Halbgruppe mit Einselement, für die eine Rechtsquotientenhalbgruppe $\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ existiert, und $\mathfrak{t}$ eine Menge bereits in $\mathfrak{N}$ invertierbarer Elemente von $\mathfrak{N}$. Dann ist auch die von $\mathfrak{n} \cup \mathfrak{t}$ erzeugte Unterhalbgruppe $\bar{\mathfrak{n}}$ rechtsseitige Nennermenge von $\mathfrak{N}$, und es gilt $\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}) = \mathfrak{Q}_r(\mathfrak{N}, \bar{\mathfrak{n}})$.*

Beweis. Ersichtlich besitzt jedes Element der von $\mathfrak{n}$ und $\mathfrak{t}$ erzeugten Unterhalbgruppe $\bar{\mathfrak{n}}$ von $\mathfrak{N}$ in $\mathfrak{S} = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ ein Inverses. Mit $\mathfrak{n} \subseteq \bar{\mathfrak{n}}$ folgt aus $\mathfrak{S} = \mathfrak{N}\mathfrak{n}^{-1}$ erst recht $\mathfrak{S} = \mathfrak{N}\bar{\mathfrak{n}}^{-1}$, womit $\mathfrak{S}$ auch Rechtsquotientenhalbgruppe von $\mathfrak{N}$ nach $\bar{\mathfrak{n}}$ ist.

Wir gehen nun von einer beliebigen Halbgruppe $\mathfrak{N} = \mathfrak{N}_0$ aus, die eine Rechtsquotientenhalbgruppe $\mathfrak{N}_1 = \mathfrak{Q}_r(\mathfrak{N}_0, \mathfrak{n}_1)$ besitzt, zu der eine Linksquotientenhalbgruppe $\mathfrak{N}_2 = \mathfrak{Q}_l(\mathfrak{N}_1, \mathfrak{n}_2)$ existieren möge u.s.f. Allgemein definieren wir induktiv:

Definition 1. Unter einer k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k$ $(k \geqq 1)$ einer Halbgruppe $\mathfrak{N} = \mathfrak{N}_0$ verstehen wir für ungerades k eine Rechtsquotientenhalbgruppe $\mathfrak{Q}_r(\mathfrak{N}_{k-1}, \mathfrak{n}_k)$, für gerades k eine Linksquotientenhalbgruppe $\mathfrak{Q}_l(\mathfrak{N}_{k-1}, \mathfrak{n}_k)$

einer $(k-1)$-ten r-Quotientenhalbgruppe $\mathfrak{N}_{k-1}$ nach einer Unterhalbgruppe $\mathfrak{n}_k$ regulärer Elemente von $\mathfrak{N}_{k-1}$. Wir schreiben dafür $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$.

Die entsprechende duale Begriffsbildung, bei der also der erste Schritt eine Linksquotientenerweiterung ist, bezeichnen wir als k-te l-Quotientenhalbgruppe.

Eine solche k-te r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$ ist dabei durch $\mathfrak{N} = \mathfrak{N}_0$ und die Nennermengen $\mathfrak{n}_\varkappa$ $(\varkappa = 1, \cdots, k)$ bis auf Isomorphie eindeutig bestimmt, da ja jeder Teilschritt durch $\mathfrak{N}_{\varkappa-1}$ und $\mathfrak{n}_\varkappa$ eindeutig festgelegt ist. Wie aus der Einleitung hervorgeht, kann man dabei ohne Beschränkung der Allgemeinheit über die Nennermengen $\mathfrak{n}_\varkappa$ noch in gewisser Weise verfügen, ohne die entstehenden Quotientenstrukturen $\mathfrak{N}_\varkappa$ abzuändern. Es empfiehlt sich jedoch nicht, diese Nennermengen in jedem Teilschritt relativ maximal zu wählen, sondern gestützt auf Hilfssatz 2 festzulegen: *Für die Nennermengen $\mathfrak{n}_1, \dots \mathfrak{n}_k$ einer k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k$ wird stets angenommen, daß für $\varkappa = 2, \cdots, k$ die Nennermenge $\mathfrak{n}_\varkappa \subseteqq \mathfrak{N}_{\varkappa-1}$ die von $\mathfrak{n}_{\varkappa-1}$ erzeugte Untergruppe von $\mathfrak{N}_{\varkappa-1}$ enthält. Mit anderen Worten, wir verabreden*

$$(*) \qquad \mathfrak{n}_{\varkappa-1} \subseteqq \mathfrak{n}_\varkappa \quad \textit{und} \quad \mathfrak{n}_{\varkappa-1}^{-1} \subseteqq \mathfrak{n}_\varkappa \qquad (\varkappa = 2, \dots, k).$$

Natürlich sind diese Nennermengen $\mathfrak{n}_\varkappa$ für allgemeine Aussagen über die k-te r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$ wenig geeignet, da die Halbgruppe $\mathfrak{n}_\varkappa$ immer erst nach dem $(\varkappa-1)$-ten Schritt zur Verfügung steht. Wir bilden daher die Durchschnitte

$$\mathfrak{x}_\varkappa = \mathfrak{n}_\varkappa \cap \mathfrak{N} \qquad (\varkappa = 1, \dots, k).$$

Diese Unterhalbgruppen $\mathfrak{x}_\varkappa$ bestehen dann aus regulären Elementen von $\mathfrak{N}$, und aus $\mathfrak{n}_1 \subseteqq \mathfrak{n}_2 \subseteqq \cdots \subseteqq \mathfrak{n}_k$ (gemäß $(*)$) folgt $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$.

Im Laufe unserer Untersuchungen wird sich in der Tat herausstellen, daß diese Kette von Unterhalbgruppen von $\mathfrak{N}$ die k-te r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$ ebenfalls eindeutig kennzeichnet und hinreichende und notwendige Kriterien für ihre Existenz auszusprechen gestattet. Wir definieren daher:

Definition 2. Eine Kette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ von Unterhalbgruppen regulärer Elemente einer Halbgruppe $\mathfrak{N}$ heißt eine Q_r-Kette von $\mathfrak{N}$ der Länge k, wenn eine k-te r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$ existiert und $\mathfrak{x}_\varkappa = \mathfrak{n}_\varkappa \cap \mathfrak{N}$ für $\varkappa = 1, 2, \cdots, k$ gilt.

Als nächstes zeigen wir zwei Hilfssätze über solche Q_r-Ketten, wobei wir dem zweiten im Hinblick auf spätere Betrachtungen eine etwas allgemeinere Fassung geben.

Hilfssatz 3. *Ist $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ eine Q_r-Kette einer Halbgruppe $\mathfrak{N}$, so gilt für $\varkappa = 2, 3, \cdots, k$ und alle Elemente a, b von $\mathfrak{N}$:*

Aus $a \cdot b \in \mathfrak{x}_\varkappa$ und $a \in \mathfrak{x}_{\varkappa-1}$ folgt $b \in \mathfrak{x}_\varkappa$; aus $a \cdot b \in \mathfrak{x}_\varkappa$ und $b \in \mathfrak{x}_{\varkappa-1}$ folgt $a \in \mathfrak{x}_\varkappa$.

Beweis. Es sei $a \cdot b \in \mathfrak{x}_\varkappa$ und $a \in \mathfrak{x}_{\varkappa-1}$. Dann folgt $ab \in \mathfrak{n}_\varkappa$ und $a \in \mathfrak{n}_{\varkappa-1}$, wobei letzteres nach $(*)$ $a^{-1} \in \mathfrak{n}_\varkappa$ nach sich zieht. Damit gilt $a^{-1}ab = b \in \mathfrak{n}_\varkappa$ und wegen $b \in \mathfrak{N}$ auch $b \in \mathfrak{x}_\varkappa = \mathfrak{n}_\varkappa \cap \mathfrak{N}$. Entsprechend folgt die zweite Behauptung.

Hilfssatz 4. *Es sei $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ eine Kette von Unterhalbgruppen regulärer Elemente einer Halbgruppe $\mathfrak{N}$, für welche die Aussagen von Hilfssatz 3 sowie $Q_r(\mathfrak{N}, \mathfrak{x}_1)$ erfüllt sind. Dann existieren die Rechtsquotientenhalbgruppen $\mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1)$ mit $\varkappa = 2, \cdots, k$ und für jedes Element $bv_1^{-1} \in \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{x}_1)$ folgt aus $bv_1^{-1} \in \mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1)$ stets $b \in \mathfrak{x}_\varkappa$.*

Beweis. Zum Nachweis der esten Behauptung zeigen wir, daß für jedes $\varkappa = 2, \cdots, k$ die Bedingung $Q_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1)$ erfüllt ist. Nun gibt es wegen $Q_r(\mathfrak{N}, \mathfrak{x}_1)$ zu Elementen $x_\varkappa \in \mathfrak{x}_\varkappa$ und $x_1 \in \mathfrak{x}_1$ Elemente $a \in \mathfrak{N}$ und $y_1 \in \mathfrak{x}_1$ mit

$$x_\varkappa y_1 = x_1 a.$$

Aus $x_1 a = x_\varkappa y_1 \in \mathfrak{x}_\varkappa$ und $x_1 \in \mathfrak{x}_1 \subseteqq \mathfrak{x}_{\varkappa-1}$ folgt aber wegen der Gültigkeit der Aussagen von Hilfssatz 3, daß das Element a sogar in $\mathfrak{x}_\varkappa$ liegt. Für die zweite Behauptung betrachten wir ein Element

$$bv_1^{-1} = x_\varkappa x_1^{-1} \in \mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1) \subseteqq \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{x}_1).$$

Dann gilt zunächst $x_\varkappa x_1^{-1} v_1 = b$. Wegen $x_1^{-1} v_1 \in \mathfrak{Q}_r(\mathfrak{x}_2, \mathfrak{x}_1)$ folgt $x_1^{-1} v_1 = t_2 t_1^{-1}$ mit $t_2 \in \mathfrak{x}_2$ und $t_1 \in \mathfrak{x}_1$. Wir erhalten dann $x_\varkappa t_2 t_1^{-1} = b$, also $x_\varkappa t_2 = bt_1 \in \mathfrak{x}_\varkappa$, woraus wegen $t_1 \in \mathfrak{x}_1 \subseteqq \mathfrak{x}_{\varkappa-1}$ nach der Aussage von Hilfssatz 3 gerade $b \in \mathfrak{x}_\varkappa$ folgt.

Für die weiteren Überlegungen ist nun der folgende Zusammenhang bedeutungsvoll, der es gestattet eine k-te Quotientenhalbgruppe auch als $(k-1)$-te Quotientenhalbgruppe aufzufassen:

Satz 1. *Es sei $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$ eine k-te r-Quotientenhalbgruppe von $\mathfrak{N}$ mit der Q_r-Kette*

$$\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k \qquad (\mathfrak{x}_\varkappa = \mathfrak{n}_\varkappa \cap \mathfrak{N}).$$

Dann ist $\mathfrak{N}_k$ zugleich die $(k-1)$-te l-Quotientenhalbgruppe $\mathfrak{Q}_l^{k-1}(\mathfrak{N}_1; \mathfrak{n}_2, \cdots, \mathfrak{n}_k)$ von $\mathfrak{N}_1 = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}_1)$ mit der Q_l-Kette

$$\mathfrak{y}_2 \subseteqq \mathfrak{y}_3 \subseteqq \cdots \subseteqq \mathfrak{y}_k \qquad (\mathfrak{y}_\varkappa = \mathfrak{n}_\varkappa \cap \mathfrak{N}_1),$$

wobei sich die $\mathfrak{y}_\varkappa$ gerade als folgende Rechtsquotientenhalbgruppen erweisen

$$\mathfrak{y}_\varkappa = \mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1) \qquad (\varkappa = 2, \ldots, k).$$

Beweis. Auf Grund unserer Definition haben wir nur die letzte Behauptung zu zeigen. Nun folgt aus $\mathfrak{N} \subseteqq \mathfrak{N}_1$ zunächst $\mathfrak{x}_\varkappa \subseteqq \mathfrak{y}_\varkappa$. Weiter gilt für $\mathfrak{x}_1 = \mathfrak{n}_1 \cap \mathfrak{N} = \mathfrak{n}_1$ nach $(*)$ gerade $\mathfrak{n}_1^{-1} \subseteqq \mathfrak{n}_2 \subseteqq \mathfrak{n}_\varkappa$ $(\varkappa = 2, \ldots, k)$, woraus mit $\mathfrak{x}_1^{-1} \subseteqq \mathfrak{N}_1$ auf

$$\mathfrak{x}_1^{-1} \subseteqq \mathfrak{N}_1 \cap \mathfrak{n}_\varkappa = \mathfrak{y}_\varkappa$$

geschlossen werden kann. Damit ist $\mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1) = \mathfrak{x}_\varkappa \mathfrak{x}_1^{-1} \subseteqq \mathfrak{y}_\varkappa$ gezeigt. Andererseits besteht $\mathfrak{y}_\varkappa$ als Unterhalbgruppe von $\mathfrak{N}_1$ aus Elementen der Form ax_1^{-1} mit $a \in \mathfrak{N}$ und $x_1 \in \mathfrak{x}_1$. Dabei liefert $\mathfrak{x}_1 \subseteqq \mathfrak{x}_\varkappa \subseteqq \mathfrak{y}_\varkappa$ sofort

$$a = ax_1^{-1} \cdot x_1 \in \mathfrak{y}_\varkappa \subseteqq \mathfrak{n}_\varkappa,$$

also $a \in \mathfrak{x}_\varkappa$ und damit $\mathfrak{y}_\varkappa \subseteqq \mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1)$.

Gestützt auf diesen Satz erhalten wir zunächst die bereits angekündigte Eindeutigkeitsaussage:

Satz 2. *Eine k-te r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_k)$ von $\mathfrak{N}$ ist durch $\mathfrak{N}$ und die Q_r-Kette*

$$\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k \qquad (\mathfrak{x}_\varkappa = \mathfrak{n}_\varkappa \cap \mathfrak{N})$$

bis auf Isomorphie eindeutig bestimmt. Wir schreiben daher im Folgenden auch

$$\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \ldots, \mathfrak{x}_k).$$

Beweis. Wir zeigen diesen Satz zusammen mit seiner dualen Aussage durch vollständige Induktion über k. Für $k=1$ ist wegen $\mathfrak{Q}_r^1(\mathfrak{N}; \mathfrak{n}_1) = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}_1)$, $\mathfrak{Q}_l^1(\mathfrak{N}; \mathfrak{n}_1) = \mathfrak{Q}_l(\mathfrak{N}, \mathfrak{n}_1)$ und $\mathfrak{n}_1 = \mathfrak{x}_1$ nichts mehr zu zeigen. Weiterhin zieht die Richtigkeit des Satzes und seiner dualen Aussage für $k-1$ die Richtigkeit beider für k nach sich, wobei wir natürlich nur einen der zueinander dualen Schritte ausführen. Gemäß Satz 1 gilt

$$\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \ldots, \mathfrak{n}_k) = \mathfrak{Q}_l^{k-1}(\mathfrak{N}_1; \mathfrak{n}_2, \ldots, \mathfrak{n}_k) \quad \text{mit} \quad \mathfrak{N}_1 = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}_1),$$

wobei die $(k-1)$-te l-Quotientenhalbgruppe nach Induktionsvoraussetzung durch $\mathfrak{N}_1$ und $\mathfrak{y}_2 = \mathfrak{n}_2 \cap \mathfrak{N}_1, \cdots, \mathfrak{y}_k = \mathfrak{n}_k \cap \mathfrak{N}_1$ bis auf Isomorphie eindeutig bestimmt ist. Nach Satz 1 gilt jedoch $\mathfrak{y}_\varkappa = \mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1)$, so daß $\mathfrak{N}_1, \mathfrak{y}_2, \cdots, \mathfrak{y}_k$ auch durch $\mathfrak{N}$ und $\mathfrak{x}_1, \cdots, \mathfrak{x}_k$ eindeutig festgelegt sind.

Nach diesen Vorbereitungen können wir uns nun der Frage nach der Existenz von k-ten r-Quotientenhalbgruppen zuwenden. Dabei wird die nachstehend definierte Bedingung $Q_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ eine wesentliche Rolle spielen:

Definition 3. Wir sagen, daß die Unterhalbgruppen $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ einer Halbgruppe $\mathfrak{N}$ der Bedingung $Q_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ genügen, wenn es zu jedem Element $z_k = z_k^1 \in \mathfrak{x}_k$ und zu jedem Element $c = c^1 \in \mathfrak{N}$ Elemente [1])

$$\begin{aligned} &z_k^j \in \mathfrak{x}_k \quad \text{und} \quad c^j \in \mathfrak{N} && (j = 2, 3, \ldots, k+1), \\ &u_j \quad \text{und} \quad v_j \quad \text{aus} \quad \mathfrak{x}_j && (j = 1, 2, \ldots, k-1) \end{aligned}$$

[1]) Da im Folgenden außer Inversen keine Potenzen auftreten werden, können wir folgende Schreibweise verabreden: Ein unterer Index i gibt an, in welcher Unterhalbgruppe $\mathfrak{x}_i$ das betreffende Element liegt, fehlt ein solcher, handelt es sich um ein beliebiges Element von $\mathfrak{N}$. Obere Indices dienen zur Unterscheidung der Elemente.

gibt, so daß folgende Gleichungen für ungerades k

$$\begin{aligned}
&c^{\lambda-1}u_{\lambda-1} = v_{\lambda-1}c^{\lambda}, \quad z_k^{\lambda-1}u_{\lambda-1} = v_{\lambda-1}z_k^{\lambda} \\
&c^{\lambda+1}u_{\lambda} = v_{\lambda}c^{\lambda}, \qquad z_k^{\lambda+1}u_{\lambda} = v_{\lambda}z_k^{\lambda},
\end{aligned} \qquad (\lambda = 2, 4, \ldots, k-1)$$
$$c^k z_k^{k+1} = z_k^k c^{k+1} \tag{1}$$

bzw. folgende Gleichungen für gerades k erfüllt sind:

$$\begin{aligned}
&c^{\lambda-1}u_{\lambda-1} = v_{\lambda-1}c^{\lambda}, \quad z_k^{\lambda-1}u_{\lambda-1} = v_{\lambda-1}z_k^{\lambda} \\
&c^{\lambda+1}u_{\lambda} = v_{\lambda}c^{\lambda}, \qquad z_k^{\lambda+1}u_{\lambda} = v_{\lambda}z_k^{\lambda}
\end{aligned} \qquad (\lambda = 2, 4, \ldots, k-2)$$
$$\begin{aligned}
c^{k-1}u_{k-1} &= v_{k-1}c^k \\
c^{k+1}z_k^k &= z_k^{k+1}c^k \\
v_{k-1}z_k^k &= z_k^{k-1}u_{k-1}.
\end{aligned} \tag{2}$$

Bei der dualen Bedingung $Q_l^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ sind natürlich die Faktoren auf beiden Seiten aller vorkommenden Gleichungen zu vertauschen.

Wir geben zunächst einige Erläuterungen zur inhaltlichen Bedeutung dieser Bedingungen: Für $k=1$ fällt offensichtlich $Q_r^1(\mathfrak{N}; \mathfrak{x}_1)$ mit der Bedingung $Q_r(\mathfrak{N}, \mathfrak{x}_1)$ zusammen. Die Bedingung $Q_r^2(\mathfrak{N}; \mathfrak{x}_1, \mathfrak{x}_2)$

$$\begin{array}{rcl}
\underline{c^1}u_1 & = & v_1 c^2 \\
 & & \;\downarrow \\
c^3 z_2^2 & = & z_2^3 c^2
\end{array}\Bigg\} \begin{array}{ll} \text{wie} & Q_l^1(\mathfrak{N}, \mathfrak{x}_2) \\ \text{auf} & c^2, z_2^2 \end{array}$$
$$\begin{array}{rcl}
\uparrow\; & & \\
v_1 z_2^2 & = & \underline{z_2^1}u_1
\end{array}$$

besagt, daß die vorgegebenen Elemente $\underline{c^1} \in \mathfrak{N}$ und $\underline{z_2^1} \in \mathfrak{x}_2$ als linke Faktoren mit den gleichen Kofaktoren $u_1, v_1 \in \mathfrak{x}_1$ in rechte Faktoren $c^2 \in \mathfrak{N}$ und $z_2^2 \in \mathfrak{x}_2$ übergeführt werden können und daß diese Elemente „weitergeschoben" werden, wobei die mittlere Gleichung der Form nach der Bedingung $Q_l^1(\mathfrak{N}, \mathfrak{x}_2)$ entspricht [2]). Analog schreiben wir die Gleichungen der Bedingung $Q_r^3(\mathfrak{N}; \mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)$ untereinander, wobei wir wieder die vorgegebenen Elemente $\underline{c^1} \in \mathfrak{N}$ und $\underline{z_3^1} \in \mathfrak{x}_3$ unterstreichen und das „Weiterschieben" der Elemente durch Pfeile kennzeichnen:

$$\begin{array}{rcl}
\underline{c^1}u_1 & = & v_1 c^2 \\
 & & \;\downarrow \\
c^3 u_2 & = & v_2 c^2 \\
\downarrow\; & & \\
c^3 z_3^4 & = & z_3^3 c^4 \\
 & & \uparrow \\
v_2 z_3^2 & = & z_3^3 u_2 \\
\uparrow\; & & \\
v_1 z_3^2 & = & \underline{z_3^1}u_1
\end{array}$$

wie $Q_r^1(\mathfrak{N}; \mathfrak{x}_3)$ auf c^3, z_3^3 (für $c^3 z_3^4 = z_3^3 c^4$); wie $Q_l^2(\mathfrak{N}; \mathfrak{x}_2, \mathfrak{x}_3)$ auf c^2, z_3^2 (für $c^3u_2 = v_2c^2$ bis $v_2 z_3^2 = z_3^3 u_2$)

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[2]) Dabei braucht die Bedingung $Q_l^1(\mathfrak{N}; \mathfrak{x}_2)$ keineswegs zu gelten.

Allgemein stellt $Q_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ eine solche Kette von Gleichungen dar, die von ihren beiden Enden her zu interpretieren ist, wobei wir freilich in Definition 3 die erste und die letzte Gleichung (letztere unter Seitenvertauschung), die zweite und die vorletzte usw. jeweils nebeneinander geschrieben haben.

Für das weitere Arbeiten empfiehlt es sich allerdings, die Schreibweise der Bedingung $Q_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ noch weiter zu formalisieren, um zu einer einfacheren Indizierung der Elemente der auftretenden Gleichungen zu gelangen und die explizite Unterscheidung der Fälle $2 \nmid k$, $2 | k$ umgehen zu können. Dazu führen wir zusätzliche Variable gemäß

$$c^1 = a^1,\; c^2 = b^1 = b^2,\; c^3 = a^3 = a^4,\; c^4 = b^3 = b^4, \ldots,$$

$$z_k^1 = x_k^1,\; z_k^2 = y_k^1 = y_k^2,\; z_k^3 = x_k^3 = x_k^4,\; z_k^4 = y_k^3 = y_k^4, \ldots$$

sowie $u_k = y_k^k$ und $v_k = x_k^k$ ein und erhalten für alle Gleichungen (1) bzw. (2) ersichtlich die einheitliche Schreibweise

$$a^\lambda u_\lambda = v_\lambda b^\lambda, \quad x_k^\lambda u_\lambda = v_\lambda y_k^\lambda \qquad (\lambda = 1, 2, \ldots, k),$$

wobei für $\lambda = k$ die triviale Gleichung $x_k^k u_k = x_k^k y_k^k = v_k y_k^k$ hinzugenommen wurde. *Zusammengefaßt können wir also die Bedingung $Q_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ auch wie folgt formulieren: Zu jedem $a^1 \in \mathfrak{N}$ und zu jedem $x_k^1 \in \mathfrak{x}_k$ gibt es Elemente*

$$a^j \in \mathfrak{N}, \quad x_k^j \in \mathfrak{x}_k \qquad (j = 2, 3, \ldots, k),$$

$$b^j \in \mathfrak{N}, \quad y_k^j \in \mathfrak{x}_k \qquad (j = 1, 2, \ldots, k),$$

$$u_j \textit{ und } v_j \textit{ aus } \mathfrak{x}_j \qquad (j = 1, 2, \ldots, k)$$

derart, daß die Gleichungen

(3) $$a^\lambda u_\lambda = v_\lambda b^\lambda, \quad x_k^\lambda u_\lambda = v_\lambda y_k^\lambda \qquad (\lambda = 1, 2, \ldots, k),$$

sowie

(4) $$v_k = x_k^k, \quad u_k = y_k^k$$

und

(5) $$a^{2i} = a^{2i+1},\; x_k^{2i} = x_k^{2i+1},\; b^{2i-1} = b^{2i},\; y_k^{2i-1} = y_k^{2i}$$

erfüllt sind, wobei (5) *stets mit $i = 1, 2, \cdots, \frac{k-1}{2}$ bzw. $i = 1, 2, \cdots, \frac{k}{2}$ für ungerades bzw. gerades k zu lesen ist* [3].

Satz 3 (Hauptsatz). *Zu einer Halbgruppe $\mathfrak{N}$ existiert eine k-te r-Quotientenhalbgruppe $\mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$*, d. h. *die Unterhalbgruppen $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ regulärer Elemente von $\mathfrak{N}$ bilden eine Q_r-Kette, genau dann, wenn die folgenden Bedingungen erfüllt sind:*

1) *Für alle Elemente a und b aus $\mathfrak{N}$ und $\varkappa = 2, 3, \cdots, k$ gilt: Aus $a \cdot b \in \mathfrak{x}_\varkappa$ und $a \in \mathfrak{x}_{\varkappa-1}$ folgt $b \in \mathfrak{x}_\varkappa$; aus $a \cdot b \in \mathfrak{x}_\varkappa$ und $b \in \mathfrak{x}_{\varkappa-1}$ folgt $a \in \mathfrak{x}_\varkappa$.*

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[3]) Im letzten Falle sehen wir die Gleichungen $a^k = a^{k+1}$ und $x_k^k = x_k^{k+1}$ als gegenstandslos an.

2) *Für jedes* $\varkappa = 1, 2, \cdots, k$ *ist die Bedingung* $Q_r^\varkappa(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_\varkappa)$ *erfüllt.*

Darüber hinaus ist $\mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ *durch* $\mathfrak{N}$ *und* $\mathfrak{x}_1 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ *bis auf Isomorphie eindeutig bestimmt.*

Beweis. Wir beweisen diesen Satz wieder zusammen mit seiner dualen Aussage, wobei die Eindeutigkeitsbehauptung bereits durch Satz 2 erledigt ist.

I. Für die Notwendigkeit der angegebenen Bedingungen haben wir wegen Hilfssatz 3 nur noch 2) nachzuweisen und schließen hier durch vollständige Induktion über k, wobei für $k = 1$ nichts mehr zu zeigen ist. Für beliebige Halbgruppen $\mathfrak{N}$ möge nun jede Q_r-Kette und jede Q_l-Kette der Länge k die Forderung 2) erfüllen. Ist nun $\mathfrak{x}_1 \subseteqq \cdots \subseteqq \mathfrak{x}_{k+1}$ eine Q_r-Kette einer Halbgruppe $\mathfrak{N}$ der Länge $k+1$, also $\mathfrak{N}_{k+1} = \mathfrak{Q}_r^{k+1}(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_{k+1})$ eine $(k+1)$-te r-Quotientenhalbgruppe von $\mathfrak{N}$, dann existiert auch die Halbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ und die Bedingungen $Q_r^\varkappa(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_\varkappa)$ für $\varkappa = 1, 2, \cdots, k$ sind nach Induktionsvoraussetzung erfüllt. Damit bleibt lediglich die Bedingung $Q_r^{k+1}(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_{k+1})$ offen. Nun ist gemäß Satz 1 die Halbgruppe $\mathfrak{N}_{k+1}$ gerade k-te l-Quotientenhalbgruppe von $\mathfrak{N}_1 = \mathfrak{Q}_r^1(\mathfrak{N}; \mathfrak{x}_1)$ nach der Q_l-Kette

$$\mathfrak{Q}_r(\mathfrak{x}_2, \mathfrak{x}_1) \subseteqq \cdots \subseteqq \mathfrak{Q}_r(\mathfrak{x}_{k+1}, \mathfrak{x}_1)$$

von $\mathfrak{N}_1$ der Länge k. Folglich gilt nach Induktionsvoraussetzung die Bedingung $Q_l^k(\mathfrak{N}_1; \mathfrak{Q}_r(\mathfrak{x}_2, \mathfrak{x}_1), \cdots, \mathfrak{Q}_r(\mathfrak{x}_{k+1}, \mathfrak{x}_1))$, d.h., zu jedem $A^1 \in \mathfrak{N}_1$ und zu jedem $X_k^1 \in \mathfrak{Q}_r(\mathfrak{x}_{k+1}, \mathfrak{x}_1)$ gibt es geeignete Elemente

$$A^j \in \mathfrak{N}_1, \quad X_k^j \in \mathfrak{Q}_r(\mathfrak{x}_{k+1}, \mathfrak{x}_1) \qquad (j = 2, 3, \ldots, k),$$
$$B^j \in \mathfrak{N}_1, \quad Y_k^j \in \mathfrak{Q}_r(\mathfrak{x}_{k+1}, \mathfrak{x}_1) \qquad (j = 1, 2, \ldots, k),$$
$$U_j \text{ und } V_j \text{ aus } \mathfrak{Q}_r(\mathfrak{x}_{j+1}, \mathfrak{x}_1) \qquad (j = 1, 2, \ldots, k),$$

die folgende Gleichungen erfüllen:

$$(3') \qquad U_\lambda A^\lambda = B^\lambda V_\lambda, \quad U_\lambda X_k^\lambda = Y_k^\lambda V_\lambda \qquad (\lambda = 1, 2, \ldots, k),$$

$$(4') \qquad V_k = X_k^k, \quad U_k = Y_k^k,$$

$$(5') \qquad A^{2i} = A^{2i+1}, \quad X_k^{2i} = X_k^{2i+1}, \quad B^{2i-1} = B^{2i}, \quad Y_k^{2i-1} = Y_k^{2i};$$

dabei gelten in (5′) die gleichen Verabredungen für den Index i wie in (5). Insbesondere trifft das für die spezielle Wahl von $A^1 = a^1$ aus $\mathfrak{N} \subseteqq \mathfrak{N}_1$ und von $X_k^1 = x_{k+1}^1$ aus $\mathfrak{x}_{k+1} \subseteqq \mathfrak{Q}_r(\mathfrak{x}_{k+1}, \mathfrak{x}_1)$ zu. Alle übrigen in (3′) auftretenden Elemente können wir als Rechtsquotienten aus $\mathfrak{N}_1 = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{x}_1)$ mit gleichem Nenner $v_1 \in \mathfrak{x}_1$

$$A^j = c^{j+1} v_1^{-1}, \quad X_k^j = z_{k+1}^{j+1} v_1^{-1} \qquad (j = 2, 3, \ldots, k),$$
$$B^j = a^{j+1} v_1^{-1}, \quad Y_k^j = x_{k+1}^{j+1} v_1^{-1} \qquad (j = 1, 2, \ldots, k),$$
$$U_j = v_{j+1} v_1^{-1}, \quad V_j = w_{j+1} v_1^{-1} \qquad (j = 1, 2, \ldots, k)$$

schreiben, wobei also $c^{j+1}\in\mathfrak{N}$, $a^{j+1}\in\mathfrak{N}$ und unter Verwendung von Hilfssatz 4 sogar $z_{k+1}^{j+1}\in\mathfrak{x}_{k+1}$, $x_{k+1}^{j+1}\in\mathfrak{x}_{k+1}$, $v_{j+1}\in\mathfrak{x}_{j+1}$ und $w_{j+1}\in\mathfrak{x}_{j+1}$ gilt. Mit dieser Darstellung erhalten wir aus (3′) für $\lambda=1$

$$B^1V_1=U_1A^1,\quad Y_k^1V_1=U_1X_k^1,$$

also

(6) $$a^2v_1^{-1}w_2=v_2v_1^{-1}a^1v_1,\quad x_{k+1}^2v_1^{-1}w_2=v_2v_1^{-1}x_{k+1}^1v_1,$$

und für $\lambda=2,3,\cdots,k$

$$B^\lambda V_\lambda=U_\lambda A^\lambda,\quad Y_k^\lambda V_\lambda=U_\lambda X_k^\lambda,$$

also

(7) $$a^{\lambda+1}v_1^{-1}w_{\lambda+1}=v_{\lambda+1}v_1^{-1}c^{\lambda+1},\quad x_{k+1}^{\lambda+1}v_1^{-1}w_{\lambda+1}=v_{\lambda+1}v_1^{-1}z_{k+1}^{\lambda+1}.$$

Die Elemente $v_1^{-1}w_2$, $v_1^{-1}a^1v_1$, $v_1^{-1}x_{k+1}^1v_1$ sowie $v_1^{-1}w_{\lambda+1}$, $v_1^{-1}c^{\lambda+1}$ und $v_1^{-1}z_{k+1}^{\lambda+1}$ von $\mathfrak{N}_1=\mathfrak{Q}_r(\mathfrak{N},\mathfrak{x}_1)$ dürfen wir wieder als Rechtsquotienten mit gleichem Nenner $t_1\in\mathfrak{x}_1$ schreiben. Gemäß Hilfssatz 4 gibt es dann zunächst Elemente $b^2\in\mathfrak{N}$, $u_2\in\mathfrak{x}_2$, $y_{k+1}^2\in\mathfrak{x}_{k+1}$ mit

$$\begin{aligned} v_1^{-1}w_2 &= u_2t_1^{-1}, \\ v_1^{-1}a^1v_1 &= b^2t_1^{-1} \qquad \text{bzw.}\qquad a^1v_1t_1 = v_1b^2, \\ v_1^{-1}x_{k+1}^1v_1 &= y_{k+1}^2t_1^{-1} \qquad \text{bzw.}\qquad x_{k+1}^1v_1t_1 = v_1y_{k+1}^2. \end{aligned}$$

Setzen wir nun noch $v_1t_1=u_1\in\mathfrak{x}_1$, so erhalten wir hieraus und aus (6)

(3_1) $$a^1u_1=v_1b^1,\quad x_{k+1}^1u_1=v_1y_{k+1}^1$$

(3_2) $$a^2u_2=v_2b^2,\quad x_{k+1}^2u_2=v_2y_{k+1}^2$$

mit $b^1=b^2$ und $y_{k+1}^1=y_{k+1}^2$ (5_1). Mit den gleichen Gedanken erhalten wir die Darstellungen

$$v_1^{-1}w_{\lambda+1}=u_{\lambda+1}t_1^{-1},\quad v_1^{-1}c^{\lambda+1}=b^{\lambda+1}t_1^{-1},\quad v_1^{-1}z_{k+1}^{\lambda+1}=y_{k+1}^{\lambda+1}t_1^{-1}\qquad(\lambda=2,\ldots,k),$$

wobei $b^{\lambda+1}\in\mathfrak{N}$, $u_{\lambda+1}\in\mathfrak{x}_{\lambda+1}$ und $y_{k+1}^{\lambda+1}\in\mathfrak{x}_{k+1}$ gilt, so daß (7) in

(3_3) $$a^{\lambda+1}u_{\lambda+1}=v_{\lambda+1}b^{\lambda+1},\quad x_{k+1}^{\lambda+1}u_{\lambda+1}=v_{\lambda+1}y_{k+1}^{\lambda+1}\qquad(\lambda=2,\ldots,k)$$

übergeht. Insgesamt haben wir dann die Aussage: Zu beliebigem $a^1\in\mathfrak{N}$ und $x_{k+1}^1\in\mathfrak{x}_{k+1}$ gibt es Elemente

$$\begin{aligned} &a^j\in\mathfrak{N},\quad x_{k+1}^j\in\mathfrak{x}_{k+1} &&(j=2,3,\ldots,k+1),\\ &b^j\in\mathfrak{N},\quad y_{k+1}^j\in\mathfrak{x}_{k+1} &&(j=1,2,\ldots,k+1),\\ &u_j \text{ und } v_j \text{ aus } \mathfrak{x}_j &&(j=1,2,\ldots,k+1), \end{aligned}$$

welche zunächst die Gleichungen (3_1), (3_2) und (3_3), also die Gleichungen

(3) $$a^\lambda u_\lambda=v_\lambda b^\lambda,\quad x_{k+1}^\lambda u_\lambda=v_\lambda y_{k+1}^\lambda\qquad(\lambda=1,2,\ldots,k+1)$$

von $Q_r^{k+1}(\mathfrak{N};\mathfrak{x}_1,\cdots,\mathfrak{x}_{k+1})$ erfüllen. Dabei gelten aber auch die weiteren Gleichungen

(4) und (5) als Folgerungen von (4′) bzw. (5′). Aus $V_k = X_k^k$, also $w_{k+1}v_1^{-1} = z_{k+1}^{k+1}v_1^{-1}$ ergibt sich zunächst $w_{k+1} = z_{k+1}^{k+1}$ und hieraus über

$$u_{k+1}t_1^{-1} = v_1^{-1}w_{k+1} = v_1^{-1}z_{k+1}^{k+1} = y_{k+1}^{k+1}t_1^{-1} \quad \text{auch} \quad u_{k+1} = y_{k+1}^{k+1},$$

während die zweite Gleichung von (4) unmittelbar aus $U_k = v_{k+1}v_1^{-1} = x_{k+1}^{k+1}v_1^{-1} = Y_k^k$ folgt. Weiter erhält man aus den Gleichungen $B^{2i-1} = B^{2i}$ bzw. $Y_k^{2i-1} = Y_k^{2i}$ von (5′) sofort $a^{2i} = a^{2i+1}$ und $x_k^{2i} = x_k^{2i+1}$, während die Gleichungen $A^{2i} = A^{2i+1}$ und $X_k^{2i} = X_k^{2i+1}$ von (5′) über $c^{2i+1} = c^{2i+2}$ und $z_{k+1}^{2i+1} = z_{k+1}^{2i+2}$ auf $b^{2i+1} = b^{2i+2}$ und $y_{k+1}^{2i+1} = y_{k+1}^{2i+2}$ führen. Zusammen mit (5_1) haben wir damit die Gleichungen

$$(5) \qquad a^{2j} = a^{2j+1}, \quad x_{k+1}^{2j} = x_{k+1}^{2j+1}, \quad b^{2j-1} = b^{2j}, \quad y_{k+1}^{2j-1} = y_{k+1}^{2j},$$

wobei der Index j in der Tat die Werte $1, 2, \cdots, \frac{k+1}{2}$ für gerades $k+1$ bzw. $1, 2, \cdots, \frac{(k+1)-1}{2}$ für ungerade $k+1$ annimmt und für gerades $k+1$ die Verabredung von Fußnote 3 zu beachten ist. Somit haben wir die Gültigkeit der Bedingung $Q_r^{k+1}(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_{k+1})$ gezeigt. Die Überlegungen für den Fall, daß $\mathfrak{N}_{k+1} = \mathfrak{Q}_l^{k+1}(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_{k+1})$ eine $(k+1)$-te l-Quotientenhalbgruppe von $\mathfrak{N}$ ist, sind zu den eben durchgeführten dual, womit dieser Teil des Beweises abgeschlossen ist.

II. Wir zeigen nun, daß die im Satz angegebenen Bedingungen 1) und 2) für die Existenz der k-ten r-Quotientenhalbgruppe $\mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ hinreichend sind, wobei wir wieder einen Induktionsschluß nach k unter Einbeziehung der dualen Behauptung durchführen. Dabei läuft für $k=1$ die Bedingung 2) gerade auf die bekannte Existenzbedingung für $\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{x}_1)$ bzw. $\mathfrak{Q}_l(\mathfrak{N}, \mathfrak{x}_1)$ hinaus. Wir nehmen daher an, daß jede Kette $\mathfrak{x}_1 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ von Unterhalbgruppen regulärer Elemente einer beliebigen Halbgruppe $\mathfrak{N}$, die 1) und 2) erfüllt, auch Q_r-Kette der Länge k von $\mathfrak{N}$ ist; entsprechend gelte die hierzu duale Aussage. Es sei nun $\mathfrak{x}_1 \subseteqq \ldots \subseteqq \mathfrak{x}_{k+1}$ eine Kette von Unterhalbgruppen regulärer Elemente einer Halbgruppe $\mathfrak{N}$ mit 1) und 2). Wegen 2) gilt zunächst $Q_r(\mathfrak{N}, \mathfrak{x}_1)$, es existiert also $\mathfrak{N}_1 = \mathfrak{Q}_r^1(\mathfrak{N}; \mathfrak{x}_1)$. Überdies genügen die Unterhalbgruppen $\mathfrak{x}_\varkappa$ $(\varkappa = 2, 3, \cdots, k+1)$ wegen 1) den Voraussetzungen von Hilfssatz 4, wonach auch die Rechtsquotientenhalbgruppen $\mathfrak{y}_\varkappa = \mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1)$ existieren. Wir werden unter a), b) und c) zeigen, daß die Unterhalbgruppen $\mathfrak{y}_2 \subseteqq \ldots \subseteqq \mathfrak{y}_{k+1}$ aus regulären Elementen von $\mathfrak{N}_1$ bestehen und die zu 1) und 2) dualen Aussagen erfüllen. Nach Induktionsvoraussetzung ist dann $\mathfrak{y}_2 \subseteqq \cdots \subseteqq \mathfrak{y}_{k+1}$ eine Q_l-Kette der Länge k von $\mathfrak{N}_1$, so daß die k-te l-Quotientenhalbgruppe $\mathfrak{Q}_l^k(\mathfrak{N}_1; \mathfrak{y}_2, \cdots, \mathfrak{y}_{k+1})$ existiert, die eine $(k+1)$-te r-Quotientenhalbgruppe $\mathfrak{Q}_r^{k+1}(\mathfrak{N}; \mathfrak{n}_1, \mathfrak{n}_2, \cdots, \mathfrak{n}_{k+1})$ von $\mathfrak{N}$ ist. Dabei gilt

$$\mathfrak{n}_1 = \mathfrak{x}_1, \quad \mathfrak{n}_2 = \mathfrak{y}_2 = \mathfrak{x}_2\mathfrak{x}_1^{-1}, \quad \mathfrak{n}_\varkappa \cap \mathfrak{N}_1 = \mathfrak{y}_\varkappa = \mathfrak{x}_\varkappa \mathfrak{x}_1^{-1} \quad \text{für} \quad \varkappa = 3, \ldots, k+1,$$

woraus zunächst

$$\mathfrak{n}_\varkappa \cap \mathfrak{N} = \mathfrak{n}_\varkappa \cap \mathfrak{N}_1 \cap \mathfrak{N} = \mathfrak{y}_\varkappa \cap \mathfrak{N} = \mathfrak{x}_\varkappa \mathfrak{x}_1^{-1} \cap \mathfrak{N} \supseteqq \mathfrak{x}_\varkappa \quad \text{für} \quad \varkappa = 2, \ldots, k+1$$

folgt. Es gilt aber auch $\mathfrak{x}_\varkappa \mathfrak{x}_1^{-1} \cap \mathfrak{N} \subseteqq \mathfrak{x}_\varkappa$, da ein Element $a = x_\varkappa x_1^{-1}$ des Durchschnitts wegen $ax_1 = x_\varkappa$ nach 1) auch $a \in \mathfrak{x}_\varkappa$ erfüllt. Damit ist aber $\mathfrak{x}_1 \subseteqq \cdots \subseteqq \mathfrak{x}_{k+1}$ die zu $\mathfrak{Q}_r^{k+1}(\mathfrak{N}; \mathfrak{n}_1, \cdots, \mathfrak{n}_{k+1}) = \mathfrak{Q}_r^{k+1}(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_{k+1})$ gehörige Q_r-Kette, also einer der zwei zueinander dualen Induktionsschlüsse von k auf $k+1$ beendet.

a) Zum Beweis der Regularität der Elemente von $\mathfrak{y}_{k+1}$ in $\mathfrak{N}_1$ sei $x_{k+1} x_1^{-1} \in \mathfrak{y}_{k+1}$ und ay_1^{-1}, by_1^{-1} seien zwei (sogleich mit dem gleichen Nenner geschriebene) Elemente aus $\mathfrak{N}_1$ mit

$$x_{k+1} x_1^{-1} a y_1^{-1} = x_{k+1} x_1^{-1} b y_1^{-1}, \quad \text{also} \quad x_{k+1} x_1^{-1} a = x_{k+1} x_1^{-1} b.$$

Da $x_1^{-1} a$ und $x_1^{-1} b$ Elemente von $\mathfrak{N}_1$ sind, können wir sie in der Form

$$x_1^{-1} a = c z_1^{-1} \quad \text{und} \quad x_1^{-1} b = d z_1^{-1}$$

schreiben, woraus zunächst $x_{k+1} c = x_{k+1} d$, also wegen der Regularität von x_{k+1} in $\mathfrak{N}$ auch $c = d$ und damit, wie behauptet,

$$a = x_1 c z_1^{-1} = x_1 d z_1^{-1} = b$$

folgt. Noch leichter ist der Nachweis für die Rechtsregularität der Elemente von $\mathfrak{y}_{k+1}$ in $\mathfrak{N}_1$.

b) Zum Nachweis der zu sich selbst dualen Aussage 1) seien $ay_1^{-1} \in \mathfrak{N}_1$ und $bz_1^{-1} \in \mathfrak{N}_1$ Elemente mit

$$ay_1^{-1} bz_1^{-1} = x_\varkappa x_1^{-1} \in \mathfrak{y}_\varkappa \quad \text{und} \quad bz_1^{-1} \in \mathfrak{y}_{\varkappa-1} \qquad (\varkappa = 3, 4, \ldots, k+1).$$

Wir führen die Multiplikation in $\mathfrak{N}_1$ aus gemäß

$$ac(z_1 t_1)^{-1} = x_\varkappa x_1^{-1} \quad \text{mit} \quad bt_1 = y_1 c, \qquad t_1 \in \mathfrak{x}_1, \quad c \in \mathfrak{N}.$$

Nun erfüllen die Halbgruppen $\mathfrak{x}_1 \subseteqq \cdots \subseteqq \mathfrak{x}_{k+1}$ die Voraussetzungen von Hilfssatz 4, wonach aus $ac(z_1 t_1)^{-1} \in \mathfrak{y}_\varkappa = \mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1)$ zunächst $ac \in \mathfrak{x}_\varkappa$ bzw. aus $bz_1^{-1} \in \mathfrak{y}_{\varkappa-1}$ auch $b \in \mathfrak{x}_{\varkappa-1}$ folgt. Aus letzterem ergibt sich wegen $ct_1^{-1} = y_1^{-1} b \in \mathfrak{Q}_r(\mathfrak{x}_{\varkappa-1}, \mathfrak{x}_1)$ auf die gleiche Weise $c \in \mathfrak{x}_{\varkappa-1}$, was zusammen mit $ac \in \mathfrak{x}_\varkappa$ zu $a \in \mathfrak{x}_\varkappa$, d.h. $ay_1^{-1} \in \mathfrak{y}_\varkappa = \mathfrak{Q}_r(\mathfrak{x}_\varkappa, \mathfrak{x}_1)$ führt. Ähnlich zeigt man, daß sich aus

$$ay_1^{-1} bz_1^{-1} = x_\varkappa x_1^{-1} \in \mathfrak{y}_\varkappa \quad \text{und} \quad ay_1^{-1} \in \mathfrak{y}_{\varkappa-1} \qquad (\varkappa = 3, 4, \ldots, k+1)$$

der Reihe nach $a \in \mathfrak{x}_{\varkappa-1}, ac \in \mathfrak{x}_\varkappa, c \in \mathfrak{x}_\varkappa, bt_1 = y_1 c \in \mathfrak{x}_\varkappa, t_1 \in \mathfrak{x}_1 \subseteqq \mathfrak{x}_{\varkappa-1}, b \in \mathfrak{x}_\varkappa$, also $bz_1^{-1} \in \mathfrak{y}_\varkappa$ ergibt.

c) Schließlich zeigen wir, daß für $\varkappa = 2, 3, \cdots, k+1$ die Bedingungen $Q_l^{\varkappa-1}(\mathfrak{N}_1; \mathfrak{y}_2, \cdots, \mathfrak{y}_\varkappa)$ erfüllt sind. Dazu seien $A^2 = a^1 x_1^{-1} \in \mathfrak{N}_1$ und $X_\varkappa^2 = x_\varkappa^1 x_1^{-1} \in \mathfrak{y}_\varkappa$ die vorgegebenen Elemente. Nach Voraussetzung gilt $Q_r^\varkappa(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_\varkappa)$, also gibt es zu $a^1 \in \mathfrak{N}$ und $x_\varkappa^1 \in \mathfrak{x}_\varkappa$ geeignete Elemente

$$a^j \in \mathfrak{N}, \quad x_\varkappa^j \in \mathfrak{x}_\varkappa \qquad (j = 2, 3, \dots, \varkappa),$$

$$b^j \in \mathfrak{N}, \quad y_\varkappa^j \in \mathfrak{x}_\varkappa \qquad (j = 1, 2, \dots, \varkappa),$$

$$u_j \text{ und } v_j \text{ aus } \mathfrak{x}_j \quad (j = 1, 2, \dots, \varkappa)$$

mit

(3) $$a^\lambda u_\lambda = v_\lambda b^\lambda, \quad x_\varkappa^\lambda u_\lambda = v_\lambda y_\varkappa^\lambda \qquad (\lambda = 1, 2, \dots, \varkappa),$$

(4) $$v_\varkappa = x_\varkappa^\varkappa, \qquad u_\varkappa = y_\varkappa^\varkappa,$$

(5) $$a^{2i} = a^{2i+1}, \quad x_\varkappa^{2i} = x_\varkappa^{2i+1}, \quad b^{2i-1} = b^{2i}, \quad y_\varkappa^{2i-1} = y_\varkappa^{2i},$$

letztere mit den entsprechenden Verabredungen für den Index i. Wir greifen aus (3) die Gleichungen für $\lambda = 1$ und $\lambda = 2$

$$a^1 u_1 = v_1 b^1, \quad x_\varkappa^1 u_1 = v_1 y_\varkappa^1$$

$$a^2 u_2 = v_2 b^2, \quad x_\varkappa^2 u_2 = v_2 y_\varkappa^2$$

heraus, wobei $b^1 = b^2$ und $y_\varkappa^1 = y_\varkappa^2$ wegen (5) gilt. Durch Rechnungen in $\mathfrak{N}_1$ erhalten wir hieraus

$$a^2 u_2 = v_2 v_1^{-1} a^1 u_1, \quad x_\varkappa^2 u_2 = v_2 v_1^{-1} x_\varkappa^1 u_1,$$

also

$$a^2 u_2 u_1^{-1} = v_2 v_1^{-1} a^1, \quad x_\varkappa^2 u_2 u_1^{-1} = v_2 v_1^{-1} x_\varkappa^1$$

und schließlich

$$a^2 u_2 (x_1 u_1)^{-1} = v_2 v_1^{-1} a^1 x_1^{-1}, \quad \varkappa^2 u_2 (x_1 u_1)^{-1} = v_2 v_1^{-1} x_\varkappa^1 x_1^{-1}.$$

Diese Gleichungen haben die Gestalt

$$B^2 V_2 = U_2 A^2, \quad Y_\varkappa^2 V_2 = U_2 X_\varkappa^2,$$

wenn wir $a^2 = B^2 \in \mathfrak{N} \subseteqq \mathfrak{N}_1$, $u_2(x_1 u_1)^{-1} = V_2 \in \mathfrak{y}_2$, $v_2 v_1^{-1} = U_2 \in \mathfrak{y}_2$, $x_\varkappa^2 = Y_\varkappa^2 \in \mathfrak{x}_\varkappa \subseteqq \mathfrak{y}_\varkappa$ setzen und $A^2 = a^1 x_1^{-1}$ und $X_\varkappa^2 = x_\varkappa^1 x_1^{-1}$ berücksichtigen. Entsprechend setzen wir allgemeiner

$$a^j = B^j \in \mathfrak{N} \subseteqq \mathfrak{N}_1, \quad x_\varkappa^j = Y_\varkappa^j \in \mathfrak{x}_\varkappa \subseteqq \mathfrak{y}_\varkappa \qquad (j = 2, 3, \dots, \varkappa),$$

$$b^j = A^j \in \mathfrak{N} \subseteqq \mathfrak{N}_1, \quad y_\varkappa^j = X_\varkappa^j \in \mathfrak{x}_\varkappa \subseteqq \mathfrak{y}_\varkappa \qquad (j = 3, 4, \dots, \varkappa),$$

$$u_j = V_j \text{ und } v_j = U_j \text{ aus } \mathfrak{x}_j \subseteqq \mathfrak{y}_j \qquad (j = 3, 4, \dots, \varkappa)$$

und erhalten somit unter Berücksichtigung auch der übrigen Gleichungen von (3): Zu vorgegebenen Elementen $A^2 \in \mathfrak{N}_1$ und $X_\varkappa^2 \in \mathfrak{y}_\varkappa$ existieren Elemente aus den entsprechenden Unterhalbgruppen von $\mathfrak{N}_1$, welche die Gleichungen

(3″) $$U_\lambda A^\lambda = B^\lambda V_\lambda, \quad U_\lambda X_\varkappa^\lambda = Y_\varkappa^\lambda V_\lambda \qquad (\lambda = 2, 3, \dots, \varkappa)$$

erfüllen. Wie man leicht nachprüft, übertragen sich aus (4) und (5) unmittelbar die (3″) ergänzenden Gleichungen (4″) und (5″), womit die Unterhalbgruppen $\mathfrak{y}_2 \subseteqq \cdots \subseteqq \mathfrak{y}_{k+1}$ von $\mathfrak{N}_1$ die zu 2) dualen Bedingungen $Q_l^{\varkappa-1}(\mathfrak{N}_1; \mathfrak{y}_2, \cdots, \mathfrak{y}_\varkappa)$ mit $\varkappa = 2, 3, \cdots, k+1$ erfüllen.

§ 3

Wir wenden uns nun einer näheren Untersuchung der Elemente einer k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ zu, die ja zunächst nur in der recht unübersichtlichen Form

$$k = 2: \ (x_2 y_1^{-1})^{-1}(a x_1^{-1})$$

$$k = 3: \ (x_2 y_1^{-1})^{-1}(a x_1^{-1})[(y_2 z_1^{-1})^{-1}(x_3 v_1^{-1})]^{-1}$$

$$\vdots$$

mit Elementen $a \in \mathfrak{N}$, $x_1 \in \mathfrak{x}_1$, $x_2 \in \mathfrak{x}_2$, $y_1 \in \mathfrak{x}_1$; $x_3 \in \mathfrak{x}_3$, $v_1 \in \mathfrak{x}_1$, $y_2 \in \mathfrak{x}_2$, $z_1 \in \mathfrak{x}_1$; $\cdots$ geschrieben werden können, wobei insgesamt 2^k Elemente von $\mathfrak{N}$ auftreten.

Satz 4. *Es sei $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ eine k-te r-Quotientenhalbgruppe einer Halbgruppe $\mathfrak{N}$ nach der Q_r-Kette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$. Dann besitzt jedes Element $a_k \in \mathfrak{N}_k$ eine Quotientendarstellung der Form*

$$a_k = x_1 x_2^{-1} x_3 \cdots x_{k-1}^{-1} \cdot a x_k^{-1} \cdot x_{k-1} \cdots x_3^{-1} x_2 x_1^{-1}, \quad k \text{ ungerade},$$

bzw.

$$a_k = x_1 x_2^{-1} x_3 \cdots x_{k-1} \cdot x_k^{-1} a \cdot x_{k-1}^{-1} \cdots x_3^{-1} x_2 x_1^{-1}, \quad k \text{ gerade},$$

mit geeigneten Elementen $a \in \mathfrak{N}$, $x_1 \in \mathfrak{x}_1, \cdots, x_k \in \mathfrak{x}_k$.

Insbesondere können wir damit $\mathfrak{N}_k$ selbst als Komplexprodukt

$$\mathfrak{N}_k = \mathfrak{x}_1 \mathfrak{x}_2^{-1} \mathfrak{x}_3 \cdots \mathfrak{x}_{k-1}^{-1} \cdot \mathfrak{N} \mathfrak{x}_k^{-1} \cdot \mathfrak{x}_{k-1} \cdots \mathfrak{x}_3^{-1} \mathfrak{x}_2 \mathfrak{x}_1^{-1}, \quad k \text{ ungerade},$$

bzw.

$$\mathfrak{N}_k = \mathfrak{x}_1 \mathfrak{x}_2^{-1} \mathfrak{x}_3 \cdots \mathfrak{x}_{k-1} \cdot \mathfrak{x}_k^{-1} \mathfrak{N} \cdot \mathfrak{x}_{k-1}^{-1} \cdots \mathfrak{x}_3^{-1} \mathfrak{x}_2 \mathfrak{x}_1^{-1}, \quad k \text{ gerade},$$

schreiben, was jedoch weniger aussagt, da für jedes Element $a_k \in \mathfrak{N}_k$ immer der gleiche „Nenner" $x_\varkappa$ bei $\mathfrak{x}_\varkappa$ und $\mathfrak{x}_\varkappa^{-1}$ verwendet werden kann. Zur besseren Übersichtlichkeit geben wir auch die entsprechende Darstellung eines Elementes a_k einer k-ten l-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_l^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ an:

$$a_k = x_1^{-1} x_2 x_3^{-1} \cdots x_{k-1} \cdot x_k^{-1} a \cdot x_{k-1}^{-1} \cdots x_3 x_2^{-1} x_1, \quad k \text{ ungerade},$$

bzw.

$$a_k = x_1^{-1} x_2 x_3^{-1} \cdots x_{k-1}^{-1} \cdot a x_k^{-1} \cdot x_{k-1} \cdots x_3 x_2^{-1} x_1, \quad k \text{ gerade}.$$

Beweis. Da der Satz und seine duale Aussage für $k = 1$ ersichtlich richtig sind, genügt es wieder, die im Satz angegebene Darstellung der Elemente a_k der k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ von $\mathfrak{N}$ aus der Darstellung herzuleiten, die nach Induktionsvoraussetzung vorliegt, wenn man $\mathfrak{N}_k$ gemäß Satz 1 als $(k-1)$-te l-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_l^{k-1}(\mathfrak{N}_1; \mathfrak{y}_2, \cdots, \mathfrak{y}_k)$ von $\mathfrak{N}_1 = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{x}_1)$ auffaßt. Ist k gerade, also $k-1$ ungerade, so gilt dann

$$a_k = Y_2^{-1} Y_3 \cdots Y_{k-1} \cdot Y_k^{-1} A \cdot Y_{k-1}^{-1} \cdots Y_3^{-1} Y_2$$

mit geeigneten Elementen $A \in \mathfrak{N}_1$, $Y_\varkappa \in \mathfrak{y}_\varkappa$ für $\varkappa = 2, 3, \cdots, k$. Dabei dürfen wir alle diese Elemente von $\mathfrak{N}_1 = \mathfrak{Q}_r(\mathfrak{N}, \mathfrak{x}_1)$ wieder mit dem gleichen Nenner $x_1 \in \mathfrak{x}_1$ schreiben,

$$A = a x_1^{-1}, \quad Y_\varkappa = x_\varkappa x_1^{-1} \qquad (\varkappa = 2, 3, \ldots, k),$$

wobei a und zunächst auch die $x_\varkappa$ Elemente von $\mathfrak{N}$ sind, jedoch nach Hilfssatz 4 sogar $x_\varkappa \in \mathfrak{x}_k$ gilt. Aus

$$a_k = (x_2 x^{-1})^{-1} x_3 x_1^{-1} \cdots x_{k-1} x_1^{-1} (x_k x_1^{-1})^{-1} a x_1^{-1} (x_{k-1} x_1^{-1})^{-1} \cdots (x_3 x_1^{-1})^{-1} x_2 x_1^{-1}$$

folgt damit die behauptete Darstellung

$$a_k = x_1 x_2^{-1} x_3 \ldots x_{k-1} \cdot x_k^{-1} a \cdot x_{k-1}^{-1} \cdots x_3^{-1} x_2 x_1^{-1}.$$

Für ungerades k, also gerades $k-1$, führt

$$a_k = Y_2^{-1} Y_3 \cdots Y_{k-1}^{-1} \cdot A Y_k^{-1} \cdot Y_{k-1} \cdots Y_3^{-1} Y_2$$

mit $A = a x_1^{-1}$ und $Y_\varkappa = x_\varkappa x_1^{-1}$ entsprechend auf

$$a_k = x_1 x_2^{-1} x_3 \cdots x_{k-1}^{-1} \cdot a x_k^{-1} \cdot x_{k-1} \cdots x_3^{-1} x_2 x_1^{-1}.$$

Insbesondere folgt aus diesem Satz, daß jede Unterhalbgruppe $\mathfrak{m}$ der k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ von $\mathfrak{N}$, die sowohl $\mathfrak{x}_k$ wie auch $\mathfrak{x}_{k-1}^{-1}$ umfaßt, eine Darstellung als Komplexprodukt

$$\mathfrak{m} = \mathfrak{x}_1 \mathfrak{x}_2^{-1} \cdots \mathfrak{x}_{k-1}^{-1} (\mathfrak{m} \cap \mathfrak{N}) \mathfrak{x}_k^{-1} \mathfrak{x}_{k-1} \cdots \mathfrak{x}_2 \mathfrak{x}_1^{-1}, \quad k \text{ ungerade},$$

bzw.

$$\mathfrak{m} = \mathfrak{x}_1 \mathfrak{x}_2^{-1} \cdots \mathfrak{x}_{k-1} \mathfrak{x}_k^{-1} (\mathfrak{m} \cap \mathfrak{N}) \mathfrak{x}_{k-1}^{-1} \cdots \mathfrak{x}_2 \mathfrak{x}_1^{-1}, \quad k \text{ gerade},$$

gestattet. Eine solche Unterhalbgruppe $\mathfrak{m}$ von $\mathfrak{N}_k$ ist also bereits durch ihren Durchschnitt $\mathfrak{m} \cap \mathfrak{N}$ mit der Halbgruppe $\mathfrak{N}$ eindeutig festgelegt, was speziell etwa für die in jedem Erweiterungsschritt verwendeten Nennermengen zutrifft.

Auch die wichtige Aussage, daß sich jeweils endlich viele Elemente einer Quotientenhalbgruppe $\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ bzw. $\mathfrak{Q}_l(\mathfrak{N}, \mathfrak{n})$ mit einem gleichen Nenner schreiben lassen, überträgt sich auf k-te Quotientenhalbgruppen:

Satz 5. *Endlich viele Elemente $a_k, b_k, \cdots$ einer k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ lassen sich stets in der Form*

$$a_k = x_1 x_2^{-1} x_3 \cdots x_{k-1}^{-1} \cdot a x_k^{-1} \cdot x_{k-1} \cdots x_3^{-1} x_2 x_1^{-1}$$

$$b_k = x_1 x_2^{-1} x_3 \cdots x_{k-1}^{-1} \cdot b x_k^{-1} \cdot x_{k-1} \cdots x_3^{-1} x_2 x_1^{-1} \qquad (k \text{ ungerade})$$

. . . .

bzw.

$$a_k = x_1 x_2^{-1} x_3 \cdots x_{k-1} \cdot x_k^{-1} a \cdot x_{k-1}^{-1} \cdots x_3^{-1} x_2 x_1^{-1}$$

$$b_k = x_1 x_2^{-1} x_3 \cdots x_{k-1} \cdot x_k^{-1} b \cdot x_{k-1}^{-1} \cdots x_3^{-1} x_2 x_1^{-1} \qquad (k \text{ gerade})$$

. . . .

mit Elementen $a, b, \cdots$ aus $\mathfrak{N}$ und dem gleichen Nenner $x_\varkappa \in \mathfrak{x}_\varkappa$ $(\varkappa = 1, 2, \cdots, k)$ schreiben.

Der Beweis erfolgt genau wie der von Satz 4, da man dort ebensogut mehrere Elemente $a_k, b_k, \cdots$ von

$$\mathfrak{R}_k = \mathfrak{Q}_r^k(\mathfrak{R}; \mathfrak{x}_1 \ldots, \mathfrak{x}_k) = \mathfrak{Q}_l^{k-1}(\mathfrak{R}_1; \mathfrak{y}_2, \ldots, \mathfrak{y}_k)$$

betrachten kann.

Wir bemerken noch, daß sich damit eine direkte Zurückführung der Addition in einem k-ten r-Quotientenhalbring $\mathfrak{R}_k = \mathfrak{Q}_r^k(\mathfrak{R}; \mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ des Halbringes $\mathfrak{R}$ (vgl. das Ende der Einleitung) auf die Addition in $\mathfrak{R}$ ergibt. Schreibt man nämlich (etwa bei ungeradem k) die Elemente a_k und b_k von $\mathfrak{R}_k$ in der in Satz 5 angegebenen Form, so gilt ersichtlich

$$a_k + b_k = x_1 x_2^{-1} x_3 \cdots x_{k-1}^{-1} \cdot (a+b) x_k^{-1} \cdot x_{k-1} \cdots x_3^{-1} x_2 x_1^{-1}.$$

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Bibliographie

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Die erste Auflage des vorliegenden Buches, in dem zum erstenmal die Systematisierung der Kristallen von rein mathematischem Standpunkt aus gegeben wurde, ist im Jahre 1947 erschienen und hat in verschiedenen Leserkreisen einen guten Empfang bekommen. Diese zweite Auflage ist als Folge des regen Interesses für dieses Buch zu betrachten. An einigen Stellen enthält sie Änderungen bzw. Korrektionen und Erweiterungen. Ganz neu wurde die Darstellung der Bewegungsgruppe des triklinen, rhomboedrischen, hexagonalen und monoklinen Systems bearbeitet. In diesen Systemen wurden nämlich neben den Gruppen von FEDOROV und SCHOENFLIES (oder den einfarbigen Gruppen) auch die zugehörigen zweifarbigen Gruppen betrachtet. Die 1651 zweifarbigen Gruppen des dreidimensionalen Raumes wurden von Gelehrten der russischen bzw. sowjetischen Schule mit geometrischen Methoden vollständig bestimmt.

Die neubearbeitete Auflage wurde mit einigen neuen Tafeln und auch mit der neuesten Literatur ergänzt.

J. Szendrei (Szeged)

Charles B. Morrey, Jr., Multiple integrals in the calculus of variations, VI+506 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1966.

Das vorliegende Buch ist von zahlreichen Gesichtspunkten von grundlegender Bedeutung unter den Monographien der Variationsrechnung. Es kann als das erste zusammenfassende Werk betrachtet werden, das diejenigen Ergebnisse systematisch behandelt, welche die Mathematiker in den letzten Jahrzehnten in Bezug auf die mehrdimensionalen Variationsprobleme erreicht haben. Das Buch strebt keine Vollständigkeit an. In den Mittelpunkt stellt es die Probleme, die theoretisch und vom Gesichtspunkt der mathematischen Anwendung her, wie die Existenz von Lösungen und das Problem der Differenzierbarkeit der Lösungen, die wichtigsten sind. In Verbindung damit werden auch die neuesten Ergebnisse, von denen mehrere dem Verfasser zu verdanken sind, bearbeitet.

Das Buch ist in zehn Kapitel aufgeteilt, deren Stoffgebiete verschiedenartig ineinandergreifen und die einzelnen Probleme miteinander verbinden.

Das erste Kapitel beginnt mit einem Überblick über die klassischen notwendigen und hinreichenden Bedingungen. Dem folgt eine kurze Behandlung der Entwicklung der direkten Methoden. Ein Teil dieses Kapitels, sowie Kapitel 3 und 4 sind der Untersuchung der Halbstetigkeit von unten gewidmet, wobei bei der Behandlung dieses Themenkreises der Schwierigkeitsgrad von Kapitel zu Kapitel gesteigert wird. Die allgemeinsten Untersuchungen beziehen sich auf die Fälle, wo die zulässigen Funktionen zu gewissen Sobolewschen Räumen gehören.

Das zweite Kapitel befasst sich mit dem Teil der Theorie über die harmonischen Funktionen und verallgemeinerten Potentiale, der im Buch zur Anwendung kommt. Schon am Ende des ersten Kapitels findet man einige Sätze, die sich auf die schwachen Lösungen der Eulerschen Gleichung beziehen. Die detaillierte Behandlung der Ergebnisse, die sich auf die Differenzierarkeit der schwachen Lösung beziehen, beinhaltet das 5. Kapitel. Dieses Kapitel beschäftigt sich unter anderem auch mit der Verallgemeinerung von Ergebnissen von DE GIORGI, NASH und MOSER und mit der Theorie von LADYSHENSKAJA und URALTZEWA.

Das 6. Kapitel behandelt die Lösungen allgemeiner elliptischer Systeme und auf die Regularität sich beziehende Untersuchungen. Die darauf bezüglichen grundlegenden Resultate von AGMON, BROWDER, DE GIORGI, DOUGLIS, F. JOHN, NASH, NIRENBERG und MORREY werden hier dargelegt.

Das 7. Kapitel zeigt Anwendungen der Variationsmethoden auf die Grundlagen der Theorie von HODGE über die harmonischen Integrale.

Das 8. Kapitel befaßt sich ebenfalls mit einer Anwendung der Variationsmethoden. Es untersucht das für äußere Differentialformen definierte ∂-Neumannsche Problem in streng pseudokonvexen komplexen analytischen Mannigfaltigkeiten.

Im 9. Kapitel wird das n-dimensionale Parameterproblem und das zweidimensionale Plateau-Problem untersucht. Es ist hervorzuheben, daß die Beweise von mehreren wichtigen bekannten Ergebnissen durch den Verfasser wesentlich vereinfacht wurden.

Das 10. Kapitel behandelt das mehrdimensionale Plateau-Problem. Auch dieses Kapitel enthält im wesentlichen die eigenen Ergebnisse der Autors, indem er die Beweise der diesbezüglichen grundlegenden Ergebnisse von REIFENBERG wesentlich vereinfacht und diese auf die Riemannschen Mannigfaltigkeiten ausdehnt.

Dieses Werk eines der hervorragendsten Experten in der Variationsrechnung bedeutet für die mathematische Literatur einen grossen Gewinn. *A. Kósa* (Budapest)

A. Blanc-Lapierre—R. Fortet, Theory of Random Functions, Volume 1, second printing, translated from French by J. Gani, xxi+443 pages, New York—London—Paris, Gordon and Breach 1967.

The book is an English translation of the French original edited in 1953, which, in spite of the several essays and books connected with the topic that have appeared since then, has retained its interest. The book aims at the unification of the demands of the pure mathematician interested in the theory itself, and those of the practical worker interested in its application. This aim is attained with no essential curbing of mathematical precision.

The present volume deals with the underlying basis in probability theory; with an introduction to the theory of random functions (from both practical and theoretical points of view); with stochastic processes, with particular respect to random functions derived from Poisson processes, which have important applications; with Markoff processes; with permanent discontinuous and continuous Markoff chains; and with additive functionals of Markoff processes. The book is supplied with an appendix on the basic mathematical notions necessary for the understanding of the subjects dealt with.

The book contains many examples important for applications. *A. Máté* (Szeged)

D. Hilbert—W. Ackermann, Grundzüge der theoretischen Logik, fifth edition, VIII+188 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1967.

The book gives accounts of propositional calculus, class-calculus, first- and second-order logic and type-theory. In this edition, which is an unchanged reprint of the fourth one, the following were the main alterations carried out in comparison with the previous editions:

The foundation of the theory of tautological truth is achieved entirely on the basis of truth-value tables, and for the axiomatization of the propositional calculus a Gentzen-type axiom system is used, which is also suitable for constructing a decision procedure. An extra account on intuitionistic logic is added.

The axiom system given for first-order logic is an extension of that given for the propositional calculus. A new chapter is adopted on the axiomatization of a predicate calculus with identity. In contrast to the earlier editions a pedantic care is here laid on the distinction between semantical and syntactical variables. The notation of quantifiers and logical connectives was changed to comply with recent standards in the literature. *A. Máté* (Szeged)

I. Singer, Cea mai bună aproximare în spaţii vectoriale normate prin elemente din subspaţii vectoriale, 386 Seiten, Bucureşti, Editura Academii Republicii Socialiste România, 1967.

Die klassische Approximationstheorie, welche auch konstruktive Funktionstheorie genannt wird, begann mit den Untersuchungen des Fragenkomplexes: Wie gut ist die Approximation einer vorgegebenen Funktion mit gewissen Polynomen, und umgekehrt, was läßt sich über eine Funktion aussagen, wenn man weiß, daß sie mit einer Polynomfolge in einer vorgegebenen Größenordnung approximiert werden kann?

Natürlich bemerkt ein aufmerksamer Leser dieser klassischen Theorie sofort, daß nicht alle Beweise an das polynomiale Charakter der approximierenden Elemente gebunden ist, und man die Güte der Approximation nicht nur mit den klassischen Mitteln, sondern auch mit den Normen recht allgemeiner linearer Verktorräume messen könnte. Versucht man aber die Tatsachen der klassischen Approximationstheorie in diese Richtung zu verallgemeinern, so entsteht von selbst eine neue Problematik: Wie weit läßt sich überhaupt das Wirkungsfeld einer nicht trivialen Approximationstheorie erweitern; in welchen Räumen existieren überhaupt bestapproximierende Elemente mit vorgeschriebenen Eigenschaften; was läßt sich über die Eindeutigkeit aussagen, usw.? Greift man diese Probleme in möglichst vollständiger Allgemeinheit an, so kommt man rasch zur Überzeugung, daß eine vertiefte Untersuchung derartiger Fragen von den Werkzeugen der Funktionalanalysis ausgiebig Gebrauch machen muß.

Das Ziel des vorliegenden Buches ist eine Monographie vorzulegen, in welcher der heutige Stand der „funktionalanalytischen" Approximationstheorie bis in die Details geschildert wird. Da zu dieser Theorie der Verfasser in zahlreichen Abhandlungen vieles beigetragen hat, ergibt sich von selbst, daß in seinem Buch in erster Linie jene Teile reichlich dargestellt wurden, welche mit dem Problemkreis des Verfassers in Zusammenhang stehen. Dieser Umstand bringt zwar eine gewisse Einseitigkeit mit sich, doch ist das kein Nachteil, da das behandelte Material eben dadurch vertieft dargestellt wird. Das Buch ist i. a. gut leserlich und gedankenerregend. Eine Übersicht des Inhalts läßt sich am besten aus den Titeln der einzelnen Kapiteln herauslesen.

I. Beste Approximation in normierten linearen Räumen mit Elementen von linearen Unterräumen. — II. Beste Approximation in normierten linearen Räumen mit Elementen von linearen Unterräumen endlicher Dimension. — III. Beste Approximation in normierten linearen Räumen mit Elementen von abgeschlossenen linearen Unterräumen endlicher Kodimension. — Appendix I. Beste Approximation in normierten linearen Räumen mit Elementen von nicht-linearen Mengen. — Appendix II. Beste Approximation in metrischen Räumen mit Elementen von beliebigen Mengen. *G. Alexits* (Budapest)

Raymond M. Smullyan, First-Order Logic (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 43), XII+158 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1968.

Reading this book does not require any particular preliminary knowledge, so it may serve for the beginner as an introduction to first-order logic; but the several new results contained in it make it interesting for the expert too.

The first part of the book is concerned with propositional calculus, with particular respect to the elegant proof procedure, the method of analytic tableaux, going back to E. W. Beth. This part of the book is concluded by the discussion of several different proofs of the compactness of the propositional calculus.

In the second part quantification theory is developed, the method of analytic tableaux is extended, the compactness property of first-order logic, the Skolem-Löwenheim theorem and the completeness theorem are proved. The Fundamental Theorem of Quantification Theory, a far-reaching extension of the completeness theorem, is established. The culmination of this part is chapter X, which gives some insight into the relation between the Lindenbaum—Henkin type completeness proofs and those of cut-free systems.

The third part of the book deals with various further topics in first-order logics such as Gentzen systems, the tableaux method for prenex formulas, the definition of a new Gentzen-type system for the derivation of CRAIG's Interpolation Lemma, symmetric completeness theorems; and finally some new systems of Linear Reasoning are considered. *A. Máté* (Szeged)

Walter Feit, Characters of finite groups (Mathematics lecture notes), VIII+186 pages, W. A. Benjamin, Inc., New York—Amsterdam, 1967.

The aim of these lecture notes is to familiarize the reader, in particular the advanced student, with some of the methods which have proved fruitful in current research in that aspect of group theory which uses the theory of characters. The representation theory of finite groups is involved in this matter with a particular emphasis on how the whole theory may be used to obtain information about the structure of finite groups.

The book starts with the definitions of representations and characters and the basic properties of these notions are developed. Next comes R. BRAUER's fundamental theorem on the character ring of a finite group, and some of the generalizations of this theorem are presented. Several applications are also given, including some concerned with splitting field and Schur indices. The remainder of the book is devoted to structural questions of finite groups such as various criteria for a group to be simple, P. HALL's characterization of solvable groups, I. G. THOMPSON's criterion for a group to have a normal p complement for an odd prime p, etc. The last part of the book contains results of mainly recent origin. Most of the proof in this part make use of the concept of a trivial intersection set which has proved useful, among other things, for studying the solvability of certain groups of odd order. Some generalization of this concept are also treated together with the related concept of coherent sets.

The book bears the imprint of the mastery the author has of his subject, and it may be expected that it will be very useful to anyone interested in this field, both as an introductory text and as a work of reference. *I. Kovács—J. Szűcs* (Szeged)

D. K. Sen, Fields and/or Particles, 138 pages, London—New York, Academic Press—Toronto, The Ryerson Press, 1968.

The problem of field-particle duality is discussed providing a brief survey of some of the fundamental classical and quantum theories of physics from an overall point of view. The author classifies the physical theories included into three categories: dualistic, non-dualistic and unified non-dualistic. From the first category he treats the classical electrodynamics and gravitation theory, and the quantum theory of particles, from the second one the classical field and particle formalism, and the quantum theory of fields, finally, from the third one, the theory of Einstein—Schrödinger, Wheeler—Misner's geometrodynamics, and Heisenberg's unified field theory. He describes the history of the attempts to understand and overcome the problem of duality, but only presents the essentials and does not attempt to go into details or give any applications. The book may serve as an introductory text for graduate students on the theories of fields and particles.

J. I. Horváth (Szeged)

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